1. a. Write the general form of the Sturm-Liouville differential equation \( Lu = 0 \) in one space dimension.

b. Write the corresponding Green’s formula (Lagrange identity) for any two solutions \( Lu = Lv = 0 \), \( a < x < b \).

c. \( L \) is called self-adjoint for the boundary conditions if \( \int_a^b (uLv - vLu) \, dx = 0 \) for any two solutions \( Lu = Lv = 0 \) satisfying the boundary conditions.

Show that the boundary conditions \( u'(a) = 0, u'(b) = -hu(b) \) lead to a self-adjoint problem.

2. Consider the second order differential equation
\[
x^2 u''(x) + 4xu'(x) + (\lambda - x^2)u(x) = 0, \quad 1 < x < 2, \quad u(1) = u(2) = 0.
\]
a. Put the equation in Sturm-Liouville form.

b. Write out the orthogonality condition for the eigenvalues.

c. Show that all eigenvalues \( \lambda > 0 \).

3. Consider the boundary value problem
\[
y''(x) + y(x) = f(x), \quad y(0) = y(\pi) = 0.
\]
a. Show that a necessary condition for a solution is that \( <f, \sin x> = 0 \).

b. Assuming the orthogonality condition of part a., find the solution by the method of eigenfunction expansion.

4a. Show that the eigenvalue problem
\[
e^{x^2} \phi'' + x \phi' + \lambda x^2 \phi = 0, \quad 1 < x < 2, \quad \phi(1) = \phi(2) = 0
\]
is a regular Sturm-Liouville eigenvalue problem and write down the orthogonality condition on the eigenfunctions.

b. It is known (see Haberman section 5.9) that for \( n \) large the large eigenvalues are asymptotically given by the formula
\[
\lambda_n \approx \left( \frac{n\pi}{\int_1^2 \sqrt{\sigma(x)p(x)} \, dx} \right)^2.
\]
Find the asymptotic value for \( \lambda_n \).

5a. Find the Green’s function for the problem
\[
u''(x) - u(x) = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0,
\]
by direct construction from \( u_1(x) = \sinh x, \quad u_2(x) = \cosh(x-1) = \cosh 1 \cosh x - \sinh 1 \sinh x. \)
b. Use the Green’s function to find the explicit solution for $f(x) = x$. Check directly that your solution is correct.

6. Consider the problem $y'' + k^2 y = f(x)$, $0 < x < \pi$, $y(0) = y(\pi) = 0$.
   a. Find the eigenfunctions and eigenvalues of $y'' + k^2 y + \lambda y = 0$, $0 < x < \pi$, $y(0) = y(\pi) = 0$.
   b. Express $f(x)$ as a Fourier series and solve the original problem. What assumptions are needed?
   c. Use your answer from part b. to immediately write down the Green’s function $G(x, x_0)$ for the original problem.