Midterm 2 Solutions

1. (25pts) Solve for $u(x, y, t)$ using the method of characteristics:

$$u_t - 2u_x + u_y = xy , \ u(x, y, 0) = 0 .$$

Be sure to check that you have found the solution. Hint: The characteristic is a straight line $(x(s), y(s), t(s))$ with $x'(s) = -2 , \ y'(s) = 1 , \ t'(s) = 1$.

The characteristic passing through $(x, y, t)$ at $s = 0$ is given by

$$x(s) = -2s + x , \ y(s) = s + y , \ t(s) = t + s ,$$

and intersects the $(x,y,0)$ plane at $s = -t$ (at the point $(x+2t,y-t,0)$). Hence

$$u(x, y, t) = \int_{-t}^{0} (x-2s)(y+s) \, ds = \int_{-t}^{0} (xy + (x-2y)s - 2s^2) \, ds$$

$$= (xys + (x-2y)s^2 - \frac{2}{3}s^3)|_{-t}^{0}$$

$$= xyt + (2y-x)t^2 - \frac{2}{3}t^3 .$$

2. (25pts) Solve by separation of variables:

$$u_{tt} - 2u = u_{xx} \text{ on } 0 < x < \pi , \ t > 0 , \ u(0,t) = u(\pi,t) = 0 , \ u(x,0) = \sin x - 2\sin 3x , \ u_t(x,0) = 0 .$$

$$u(x,t) = G(t)\phi(x)$$

gives

$$\frac{G'' - 2G}{G} = \frac{\phi''}{\phi} = -\lambda .$$

Since $\phi(0) = \phi(\pi) = 0, \ \phi(x) = \sin nx$ and $\lambda = n^2$. Hence

$$G'' + (n^2 - 2)G = 0 .$$

Thus $G(t) = A_n \cos \sqrt{n^2 - 2t}$ for $n \geq 2$ and $G(t) = A_1 \cosh t$ for $n = 1$. This gives

$$u(x,t) = A_1 \cosh t \sin x + \sum_{n=2}^{\infty} A_n \cos \sqrt{n^2 - 2t} \sin nx .$$
Since \( u(x,0) = \sin x - 2 \sin 3x \), \( A_1 = 1 \), \( A_3 = -2 \) and all other \( A_n = 0 \).

3. (25pts) Solve using D’Alembert’s formula and the method of reflection:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} \\
&= 0, \\
u(0,t) &= 0, \\
u(x,0) &= x^2, \\
u_t(x,0) &= 0.
\end{align*}
\]

Be sure to give the solution explicitly in the regions \( x > t > 0 \) and \( 0 < x < t \).

Let \( f(x) = -x^2 \) for \( x < 0 \) and \( f(x) = x^2 \) for \( x > 0 \) Then by D’Alembert,

\[
\pi(x,t) = \frac{1}{2}(f(x+t) + f(x-t)).
\]

This gives

\[
\begin{align*}
u(x,t) &= \frac{1}{2}(x^2 + t^2) \quad \text{for} \quad x > t > 0 \quad \text{and} \\
u(x,t) &= \frac{1}{2}((x+t)^2 - (x-t)^2) = 2xt \quad \text{for} \quad 0 < x < t.
\end{align*}
\]

4. (25pts) Solve using Duhamel’s principle:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + xt \\
&= 0, \\
u(x,0) &= e^x, \\
u_t(x,0) &= 0.
\end{align*}
\]

By superposition and D’Alembert, \( u(x,t) = \frac{1}{2}(e^{x+t} + e^{x-t}) + v(x,t) \) where \( v \) satisfies

\[
\begin{align*}
v_{tt} &= v_{xx} + xt, \\
v(x,0) &= v_t(x,0) = 0.
\end{align*}
\]

By Duhamel’s principle, \( v(x,t) = \int_0^t w(x,t;s) \, ds \), where

\[
\begin{align*}
w_{tt} &= w_{xx}, \\
w(x,s) &= 0, \\
w_t(x,s) &= xs.
\end{align*}
\]

Hence

\[
w(x,t;s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} sy \, dy = \frac{s}{4}[(x+(t-s))^2 - (x-(t-s))^2] = sx(t-s).
\]

This gives

\[
v(x,t) = \int_0^t xs(t-s) \, ds = x \int_0^t (ts - s^2) \, ds = x(t^3 - 3t^3)/6 = t^3/6.
\]