Review problems for Midterm 1

1. Review the solution from my first week lectures of the simple transport equation

\[ u_t + bu_x = 0 \ , \ u(x,0) = g(x) \ . \]

Now figure out how to solve (by a change of dependent variable \( u(x,t) \) goes to \( v(x,t) \))

\[ u_t + bu_x + cu = 0 \ , \ u(x,0) = g(x) \ . \]

Here \( b \) and \( c \) are constants.

2. Solve the inhomogeneous heat equation \( u_t = u_{xx} - \sin x \) on \((0, \pi)\) with

\[
\begin{align*}
    u(0,t) &= 0 \ , \ u_x(\pi,t) = 0 \ , \ u(x,0) = 2 \sin 2x - x .
\end{align*}
\]

3. A thin rectangular plate bounded by the lines \( x=0, \ x=a, \ y=0, \ y=b \) whose surface is impervious to heat flow is given an initial temperature distribution \( \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \). Its four edges are kept at zero temperature. Find the temperature distribution at later times.

4. Find the steady state temperature \( u(r, \theta) \) of a thin plate over the sector

\[ \Omega = \{(r, \theta) : 0 < r < 1 \ , \ 0 < \theta < \frac{\pi}{3} \} \]

given that \( u(r,0) = 0 \ , \ u_r(r, \frac{\pi}{3}) = 0 \) and \( u(1, \theta) = \sin \frac{3}{2} \theta \). You may assume that \( u \) is bounded.

5. Find the solution of Laplace’s equation in a disk of radius \( R \) with boundary values 1 on the upper semicircle and 0 on the lower semicircle boundaries.

6. Let \( f(x) \) be defined on the real line by

\[
    f(x) = \begin{cases} 
    2 - x & \text{if } 0 < x < 3 \\
    0 & \text{if } -3 < x < 0 \\
    f(x + 6) = f(x) & \text{otherwise}
    \end{cases}
\]

Find and plot the Fourier series of \( f \) on \((-6,6)\).

7. Find a cosine series to represent \( f(x) = e^x \) in \( 0 \leq x < \pi \). Sketch the series over the range \((-2\pi, 2\pi)\).
8. Show that in \((-\pi, \pi)\),

\[ x \sin x = 1 - \frac{1}{2} \cos x + 2 \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\cos nx}{n^2 - 1} . \]

Use this to deduce that

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} = \frac{1}{4} . \]

b. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} = \frac{\pi - 2}{4} . \]