Review problems for Midterm 2

1. Solve \( u_t + 2u_x = xe^{-t} \), \( u(x,0) = 0 \).

2. Solve by separation of variables:
   \[
   u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \quad \text{where}
   \]
   \[
   u_x(0,t) = u_x(\pi,t) = 0, \quad u(x,0) = x, \quad u_t(x,0) = 0.
   \]

3. Find \( u(\frac{1}{4}, \frac{7}{2}) \) if \( u_{tt} = u_{xx}, \quad 0 < x < 1, \quad u(x,0) = x^2(1-x), \quad u_t(x,0) = (1-x)^2 \)
   \( u(0,t) = u(1,t) = 0 \).

   Hint: Extend the initial data to (-1,0) by odd reflection and then extend the data (which is now defined on (-1,1)) to the whole line as a periodic function of period 2. Use the D’Alembert formula and evaluate the answer carefully.

4. Solve by separation of variables:
   \[
   u_{tt} + 2u_t = 4u_{xx}, \quad 0 < x < \pi, \quad t > 0, \quad \text{where}
   \]
   \[
   u(0,t) = u(\pi,t) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 1.
   \]

5. Consider the inhomogeneous heat equation:
   \[
   u_t = u_{xx} + Q(x,t), \quad 0 < x < \pi, \quad t > 0 \quad \text{with} \quad u(x,0) = u(0,t) = u(\pi,t) = 0.
   \]
   a. Solve using the method of eigenfunction expansion, that is let
   \[
   u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx, \quad Q(x,t) = \sum_{n=1}^{\infty} Q_n(t) \sin nx \quad \text{and solve for} \quad b_n(t).
   \]
   b. Solve using Duhamel’s principle, that is,
   \[
   u(x,t) = \int_0^t w(x,t;\tau) \, d\tau, \quad \text{where}
   \]
   \[
   w(x,\tau) = Q(x,\tau), \quad w(0,t) = w(\pi,t) = 0.
   \]
   Solve for \( w \) by separation of variables.
c. Show that the results in parts a,b give the same answer.

6. Solve $u_{tt} = 4u_{xx} + x \sin t$, $-\infty < x < \infty$, $t > 0$ with $u(x, 0) = u_t(x, 0) = 0$ using Duhamel’s principle. That is,

$$u(x, t) = \int_0^t w(x, t; \tau) \, d\tau,$$

where $w$ is the solution of

$$w_{tt} = 4w_{xx} \ - \infty < x < \infty, \ t > \tau \text{ with } w(x, \tau) = 0, \ w_t(x, \tau) = x \sin \tau.$$

Hint: To find $w$, use D’Alembert’s formula but be careful to replace $t$ by $t - \tau$ since we are solving for $t > \tau$. 

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