1. \[ \frac{\partial}{\partial t} \left[ e(x,t)A(x)\right] = -\frac{\partial}{\partial x} \left[ \phi(x,t)A(x) \right] \]
\[ A(x) \frac{\partial e(x,t)}{\partial t} + A(x) \frac{\partial \phi(x,t)}{\partial x} + \phi(x,t) \frac{\partial A(x)}{\partial x} = 0 \]
\[ c\rho A(x) \frac{\partial u(x,t)}{\partial t} - K_0 A(x) \frac{\partial^2 u(x,t)}{\partial x^2} - K_0 \frac{\partial u(x,t)}{\partial x} \frac{\partial A(x)}{\partial x} = 0 \]
\[ k = \frac{k_0}{c\rho} \]
\[ \therefore \frac{\partial u(x,t)}{\partial t} = k \left( \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} \frac{1}{A(x)} \right) \]

2. (a). \[ \frac{\partial u(x,t)}{\partial t} \frac{\partial u(x,t)}{\partial x} A \Delta x = (\phi(x,t) - \phi(x + \Delta x,t))A \]
\[ \frac{\partial u(x,t)}{\partial x} = -k \frac{\partial u(x,t)}{\partial x} \]

(b). \[ \int_a^b \left( \frac{\partial u(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial x} \right) dx = 0 \]
\[ \int_a^b \left( \frac{\partial u(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial x} \right) dx = 0 \]
\[ \therefore \frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2} \]

3. No heat energy is lost at \( x = x_0 \)
\[ \phi(x_0-,t) = \phi(x_0+,t) \]
\[ -K_0(x_0-) \frac{\partial u(x_0-,t)}{\partial x} = -K_0(x_0+) \frac{\partial u(x_0+,t)}{\partial x} \]
The condition of \( \frac{\partial u}{\partial x} \) continuous at \( x_0 \) is
\[ \frac{\partial u}{\partial x}(x_0-,t) = \frac{\partial u}{\partial x}(x_0+,t) \] i.e. \( K_0(x_0-) = K_0(x_0+) \)

4. \[ K_0 \frac{d^2 u}{dx^2} + Q = 0 \]
\[ \frac{d^2 u}{dx^2} + 1 = 0 \]
\[ \therefore u = -\frac{1}{2} x^2 + C_1 x + C_2 \]
\[ u(0) = T_1, u(L) = T_2 \]
\[ \therefore u = -\frac{1}{2} x^2 + \frac{T_2 - T_1 + \frac{1}{2} L^2}{L} x + T_1 \]

5. \[ \frac{d^2 u}{dx^2} = 0 \]
\[ \therefore u = C_1 x + C_2 \]
\[ u(0) = T, \frac{d u}{dx}(L) + u(L) = 0 \]
\[ \therefore u = -\frac{T}{1+L} x + T \]
6. \[0,1\] \[\rho K \frac{d^2u}{dx^2} + Q = 0\] \[\therefore \frac{d^2u}{dx^2} = -1\]
\[\therefore u = -\frac{1}{2} x^2 + C_1 x + C_2\]
\[u(0) = 0 \Rightarrow C_2 = 0\]
\[1,2\] \[\frac{d^2u}{dx^2} = 0 \therefore u = C_3 x + C_4\]
\[u(2) = 0 \Rightarrow 2C_3 + C_4 = 0\]
use the equation in problem 4, we get
\[\therefore \frac{d^2u}{dx^2} = -\frac{1}{2} x^2 + \frac{2}{3} x\] in \([0,1]\)
\[\therefore \frac{d^2u}{dx^2} = -\frac{1}{6} x + \frac{1}{3}\] in \([1,2]\)

7. \[\frac{d^2u}{dx^2} + 1 = 0 \Rightarrow u = -\frac{1}{2} x^2 + C_1 x + C_2\]
\[\frac{du}{dx}(0) = 1 \Rightarrow C_1 = 1\]
\[\frac{du}{dx}(L) = \beta \Rightarrow -L + C_1 = \beta\]
So we get \(\beta = 1 - L\)
This indicates that the rate of temperature change at \(L\) dependent on the value of \(L\).

8. \[
\frac{\partial u}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial y^2}
\]
plug (1) in this formula. \(\frac{\partial^2 u}{\partial y^2}\) is in the same way.

9. \[\nabla^2 u = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} = 0\]
Since it is circularly symmetric,
\[\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) = 0\]
\[u(r_1) = T_1 \Rightarrow C_1 \ln r_1 + C_2 = T_1\]
\[u(r_2) = T_2 \Rightarrow C_1 \ln r_2 + C_2 = T_2\]
\[C_1 = \frac{T_2 - T_1}{\ln (r_2/r_1)}\]
\[C_2 = \frac{T_1 \ln (r_2/r_1)}{\ln (r_2/r_1)}\]
\[u = \frac{T_1 \ln (r_2/r_1) + T_2 \ln (r_2/r_1)}{\ln (r_2/r_1)}\]

10. When \(t \to \infty\), we have \(\frac{k}{r} \frac{d}{dr} (r \frac{du}{dr}) = 0\)
\[u = C_1 \ln r + C_2\]
\[\frac{du}{dr}(a) = \beta \Rightarrow \frac{C_1}{a} = \beta\]
\[\frac{du}{dr}(b) = \beta \Rightarrow C_1 = b\]
\[\therefore \beta = \frac{b}{a}\]
The rate of temperature change at the inner radius is dependent on the value of \(\frac{b}{a}\).