1. Since \( f \) is measurable, \( \int_E f \) exists. Since all the \( E_k \) are disjoint, by theorem 5.24 we have
\[
\int_E f = \sum_k \int_{E_k} f = \sum_k \int_{E_k} f_{E_k} = \sum_k \int_{E_k} a_k = \sum_k a_k |E_k|.
\]

2. If \( f_k = -\chi_{(k, \infty)} \), then all the \( f_k \) are measurable, bounded below by \( \phi : \mathbb{R} \to \mathbb{R} \) defined by \( \phi(x) = -1 \), and they increase to \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 0 \), but \( \int \chi_{(k, \infty)} = -\infty \) does not converge to \( \int 0 = 0 \). If \( f_k = \chi_{(k, \infty)} \), then all the \( f_k \) are measurable, bounded above by \( \phi : \mathbb{R} \to \mathbb{R} \) defined by \( \phi(x) = 1 \), and they decrease to \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 0 \), but \( \int \chi_{(k, \infty)} = \infty \) does not converge to \( \int 0 = 0 \).

3. Since each \( f_k \) is measurable, \( f \) is measurable. Since \( f_k \leq \phi \) almost everywhere, where \( \phi = f \), if \( \int_E f \) is finite then by theorem 5.19 \( \int_{E_k} f_k \to \int_E f \). If \( \int_E f = \infty \), then the fact that \( f_k \to f \) implies that \( \int_{E_k} f_k \to \infty \).

4. Since \( |x^k f(x)| \leq |f(x)| \) for all \( x \in [0,1] \), by theorem 5.10 \( \int_{[0,1]} |x^k f(x)| \leq \int_{[0,1]} |f(x)| \). Since \( f \in L([0,1]) \), by theorem 5.21 \( |f| \in L([0,1]) \), so \( |x^k f(x)| \in L([0,1]) \) and again applying theorem 5.21 we have \( x^k f(x) \in L([0,1]) \). Since \( f \in L([0,1]) \), by theorem 5.22 \( f \) is finite almost everywhere on \([0,1] \), so \( x^k f(x) \to 0 \) almost everywhere on \([0,1] \). Since \( |x^k f(x)| \leq \phi(x) \) for all \( x \in [0,1] \), where \( \phi = |f| \in L([0,1]) \), by theorem 5.36 \( \int_{[0,1]} x^k f(x) \to \int_{[0,1]} 0 = 0 \).

5. By Egorov’s theorem, there exists a closed set \( F_1 \subset E \) such that \( |E - F_1| < 2^{-1}|E| \) and \( f_k \) converges uniformly to \( f \) on \( F_1 \), and for any \( n \) there exists a closed set \( F_{n+1} \subset E - \bigcup_{i=1}^{n} F_i \) such that
\[
\left| E - \bigcup_{i=1}^{n} F_i \right| - F_{n+1} = \left| E - \bigcup_{i=1}^{n+1} F_i \right| < 2^{-n+1}|E| \quad \text{and} \quad f_k \to f \quad \text{uniformly on} \quad F_{n+1}.
\]
Then \( E - \bigcup_{i=1}^{n} F_i \subset E - \bigcup_{i=1}^{n+1} F_i \) for each \( n \), so \( E - \bigcup_{i=1}^{n} F_i \subset E - \bigcup_{i=1}^{n} F_i \subset E - \bigcup_{i=1}^{n} F_i \subset E - \bigcup_{i=1}^{n} F_i \) for each \( n \), so
\[
\int_{E - \bigcup_{i=1}^{n} F_i} f_k = 0 \quad \text{and} \quad \int_{E - \bigcup_{i=1}^{n} F_i} f = 0.
\]
Since each \( F_i \) is closed, \( \bigcup_{i=1}^{n} F_i \) is measurable, so by theorem
5.24 \( \int_{E} f = \int_{F_1} f + \int_{F_2} f = \int_{F_k} f \). Since all the \( F_i \) are measurable and disjoint, again applying theorem 5.24 we have 
\[ \int_{E} f_k = \sum_{i=1}^{\infty} f_k \]
so \( \lim_{k \to \infty} \int_{E} f_k = \lim_{k \to \infty} \sum_{i=1}^{\infty} f_k = \lim_{n \to \infty} \sum_{i=1}^{n} f_k \). Since 
\[ |f_k| \leq M \quad \text{and} \quad |F_i| \leq |E - F_{i-1}| < 2^{-i}|E| \quad \text{for} \quad i \geq 1 \quad (\text{with} \quad F_0 = \emptyset) \], we have 
\[ \left| \int_{F_i} f_k \right| \leq 2^{-i}|E|M \quad \text{and} \quad \sum_{i=1}^{\infty} 2^{-i}|E|M = 2|E|M < \infty \]. Thus by the Weierstrass M-test, the rate at which \( \sum_{i=1}^{n} f_k \) converges to \( \sum_{i=1}^{\infty} f_k \) as \( n \to \infty \) is independent of \( k \). Thus we can interchange the two limits in our expression for \( \lim_{k \to \infty} \int f_k \), so we get 
\[ \lim_{k \to \infty} \int_{E} f_k = \lim_{k \to \infty} \sum_{i=1}^{n} \int_{F_i} f_k = \lim_{n \to \infty} \sum_{i=1}^{n} \int_{F_i} f_k = \lim_{n \to \infty} \sum_{i=1}^{n} \int_{F_i} f_k \].
Since \( f_k \) converges uniformly to \( f \) on each \( F_i \) and \( |F_i| < \infty \), by theorem 5.33, \( \lim_{n \to \infty} \int_{F_i} f_k = \int_{F_i} f \), so we have 
\[ \lim_{k \to \infty} \int_{F_i} f_k = \lim_{n \to \infty} \sum_{i=1}^{n} \int_{F_i} f_k = \lim_{n \to \infty} \sum_{i=1}^{n} \int_{F_i} f = \sum_{i=1}^{\infty} \int_{F_i} f = \sum_{i=1}^{\infty} \int_{F_i} f = \int_{E} f . \]

6. We have 
\[ \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{1}{h} \left( f(x + \frac{1}{n}, y) - f(x, y) \right) \]
\[ = \lim_{n \to \infty} \left( f(x + \frac{1}{n}, y) - f(x, y) \right) \]. Since \( f \) is a measurable function of \( y \), so is \( f(x + \frac{1}{n}, y) \), and thus so is \( n \left( f(x + \frac{1}{n}, y) - f(x, y) \right) \), which implies that 
\[ \lim_{n \to \infty} n \left( f(x + \frac{1}{n}, y) - f(x, y) \right) = \frac{\partial f}{\partial x} \] is a measurable function of \( y \). Since \( \frac{\partial f}{\partial x} \) is bounded, let \( M = \sup_{[0,1] \times [0,1]} \left| \frac{\partial f}{\partial x} \right| \). Since 
\[ \lim_{n \to \infty} n \left( f(x + \frac{1}{n}, y) - f(x, y) \right) = \frac{\partial f}{\partial x} \], it follows that 
\[ \lim_{n \to \infty} \left| n \left( f(x + \frac{1}{n}, y) - f(x, y) \right) \right| = \left| \frac{\partial f}{\partial x} \right| \], so 
\[ n \left( f(x + \frac{1}{n}, y) - f(x, y) \right) \] is eventually bounded by \( M + \varepsilon \) for any \( \varepsilon \). By corollary 5.37, this implies that 
\[ \int_{[0,1]} n \left( f(x + \frac{1}{n}, y) - f(x, y) \right) dx \to \int_{[0,1]} \frac{\partial f}{\partial x} \] .

7. We have 
\[ \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi < \infty \], but 
\[ \int_{-\infty}^{\infty} \left| \sin x \right| dx = \infty \], so 
\[ \left| \sin x \right| \notin L(\mathbb{R}) \] and thus 
\[ \frac{\sin x}{x} \notin L(\mathbb{R}) \].