Real Variables: Solutions to Homework 9

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Exercise 0.1. Chapter 8, # 1: For complex-valued, measurable $f$, $f = f_1 + i f_2$ with $f_i$ real-valued and measurable, we have $\int_E f = \int_E f_1 + i \int_E f_2$. Prove that $\int_E f$ is finite if and only if $\int_E |f|$ is finite, and $|\int_E f| \leq \int_E |f|$.

Proof. Firstly, we know by Theorem (5.21) that if $f$ is measurable, $f \in L^1$ if and only if $|f| \in L^1$ so both integrals or either finite or infinite. Now, for some $\theta \in \mathbb{R}$, write

$$\int_E f = \left| \int_E f \right| e^{i\theta}.$$

Then, calculate

$$\left| \int_E f \right| = e^{-i\theta} \int_E f = \int_E e^{-i\theta} f$$

$$= Re \int_E e^{-i\theta} f = \int_E Re (e^{-i\theta} f)$$

$$\leq \int_E |e^{-i\theta} f| = \int_E |f|.$$

Here we have use the fact that the integral is real to proceed from the first line to the second line and we used the fact that $|e^{-i\theta}| = 1$ to get the last equality.

Exercise 0.2. Chapter 8, # 2: Prove the converse of Holder’s inequality for $p = 1$ and $\infty$. Show also that for real-valued $f \notin L^p(E)$, there exists a function $g \in L^{p'}(E)$, $1/p + 1/p' = 1$, such that $fg \notin L^1(E)$.

Proof. First for $p = 1$. Then it is clear that for all $g \in L^\infty$ with $\|g\|_\infty \leq 1$,

$$\|f\|_1 \geq \sup \int_E f g.$$

For the other direction, take $g = \text{sgn} f$, then $g \in L^\infty$ and $\|g\|_\infty \leq 1$.

$$\|f\|_1 = \int_E |f| = \int_E f g.$$

But this gives us the other direction for the inequality and therefore we have proven what we set out to prove.
Now for $p = \infty$, the for all $g \in L^1$ with $\|g\|_1 \leq 1$, then we have

$$\|f\|_{\infty} \geq \sup_E fg.$$  

Again, for the other direction, there are 3 cases, $\|f\|_{\infty} = 0$, $\|f\|_{\infty}$ finite and $\|f\|_{\infty} = \infty$. For $\|f\|_{\infty} = 0$ it is immediate. For $\|f\|_{\infty}$ finite, suppose without loss of generality $\|f\|_{\infty} = 1$. Define set $E := \{ x \in E : |f(x)| > 1 - \frac{1}{n} \text{ with } n \in \mathbb{N} \}$. The measure $|E_n| > 0$ for all $n$ and define function $g$ such that

$$g_n(x) = \begin{cases} \text{strictly positive with } & \int_{E_n} g_n(x) = 1 \quad x \in E_n \\ 0 & \text{otherwise} \end{cases}.$$  

Then

$$\int_E |f| g_n = \int_{E_n} |f| g_n \geq \left( 1 - \frac{1}{n} \right) \int_{E_n} g_n = 1 - \frac{1}{n}.$$  

Then for all $g \in L^1$ with $\|g\|_1 \leq 1$, then

$$\|f\|_{\infty} = \sup_E \int_E |f| g = \sup_E \int_E fg.$$  

For the last case, repeat the above argument with the set $F := \{ x \in E : |f(x)| > n \text{ with } n \in \mathbb{N} \}$ and the result follows. 

**Exercise 0.3.** Chapter 8, # 3: Prove theorems (8.12) and (8.13). Show that Minkowski’s inequality for series fails when $p < 1$. We will simplify the statements in the text by proving the result for functions $f$ and $g$.

**Theorem 0.4.** *Holder’s Inequality for sequences.*

**Proof.** We will prove Holder’s inequality for sequences by employing the more general statement for functions from the text. The proof the the statement (Theorem 8.6) can be found in the text. Take the $f$ and $g$ given in the theorem to be step functions defined in the following way:

$$f(x) = \begin{cases} a_k & x \in (k - 1, k] \text{ for } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$  

$$g(x) = \begin{cases} b_k & x \in (k - 1, k] \text{ for } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$  

Then it is clear that

$$\int_E |fg| = \sum_{k=1}^{\infty} |a_k b_k|,$$

$$\|f\|_{L^p} = \|a_k\|_{\ell^p} \text{ and } \|g\|_{L^p} = \|b_k\|_{\ell^p}.$$  

Thus the theorem is proven by substitution into Theorem 8.6. 

\[ \square \]
Theorem 0.5. Minkowski’s inequality for sequences.

Proof. Minkowski’s inequality is proven in precisely an analogous way to the way we proved Holder’s inequality by taking \( f \) and \( g \) given in the theorem to be step functions defined in the same way and then applying Theorem 8.13.

However, feeling like this was not enough work, here is an alternate proof of it. Let \( \frac{1}{q} = 1 - \frac{1}{p} \). Now we know by Holder that

\[
\sum_{i=1}^{n} |a_i + b_i|^{p/q} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/q}
\]

also

\[
\sum_{i=1}^{n} |a_i + b_i|^{p/q} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/q}.
\]

Because \( p = 1 + \frac{p}{q} \), we know that

\[
|a_i + b_i|^p = |a_i + b_i||a_i + b_i|^{p/q}
\]

and putting together the two above statements, we have

\[
\sum_{i=1}^{n} |a_i + b_i|^{p/q} \leq \left[ \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/q} \right] \left( \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/q}
\]

but \( 1/p + 1/q = 1 \) so

\[
\left( \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left[ \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/q} \right].
\]

This is Minkowski’s inequality.

\[ \square \]

To see that Minkowski’s inequality does not hold for \( p < 1 \), Take \( a = (1, 0, 0, \cdots) \) and \( b = (0, 1, 9, 0, \cdots) \).

Exercise 0.6. Chapter 8, # 4: Let \( f \) and \( g \) be real-valued, and let \( 1 < p < \infty \). Prove that equality holds in the inequality \( |\int fg| \leq \|f\|_p \|g\|_{p'} \) if and only if \( fg \) has constant sign a.e. and \( |f|^p \) is a multiple of \( |g|^{p'} \) a.e.

Proof. First the easier direction. Let \( |f|^p = c|g|^q \). Then

\[
\|f\|_p \|g\|_q = \left( \int_E |f|^p \right)^{1/p} \left( \int_E |g|^q \right)^{1/q} = c^{1/p} \left( \int_E |g|^q \right)^{\frac{q-p}{q}} \left( \int_E |g|^q \right)^{1/q} = c^{1/p} \left( \int_E |g|^q \right).
\]
But $|f||g| = c^{1/p}|g|^{q-1}|g| = c^{1/p}|g|^q$ so we have the desired result.

For the other direction, normalize the functions so that

$$F = \frac{|f|}{\|f\|_p} \quad \text{and} \quad G = \frac{|g|}{\|g\|_q}.$$  

Then the $p$ norms of $F$ and $G$ are 1. Supposing Holder’s equality holds, then

$$\int_E FG = 1 = \frac{1}{p} F^p + \frac{1}{q} \int G^q.$$  

But $FG < \frac{1}{p} F^p + \frac{1}{q} G^q$ by Young’s inequality. But then

$$\int_E \left( \frac{1}{p} F^p + \frac{1}{q} G^q - FG \right) = 0 \quad \text{so} \quad \frac{1}{p} F^p + \frac{1}{q} G^q - FG = 0 \ a.e.$$  

This happens if and only if $F^p = G^q \ a.e.$ so

$$|f|^p = \frac{\|f\|_p}{\|g\|_q} |g|^q.$$  

With that we are done. \hfill \Box

**Exercise 0.7.** Chapter 8, # 5: For $1 < p < \infty$ and $0 < |E| < \infty$, define

$$N_p[f] = \left( \frac{1}{|E|} \int_E |f|^p \right)^{1/p}.$$  

Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$. Prove also that $N_p[f + g] \leq N_p[f] + N_p[g]$, $\int_E |fg| \leq N_p[f] N_p[g]$, $1/p + 1/p' = 1$, and that $\lim_{p \to \infty} N_p[f] = \|f\|_\infty$.

**Proof.** If $p_1 < p_2 < \infty$, let $q = \frac{p_2}{p_1} > 1$. Then, solving for the conjugate exponent

$$\frac{1}{q} + \frac{1}{x} = 1$$

we find that $x = \frac{q}{q-1} = \frac{p_2}{p_2-p_1} > 1$. Now we estimate with Holder’s inequality:

$$\|f\|_{p_1} = \int_E |f|^{p_1} \leq \int_E |f|^{p_1} \times 1 \leq \left( \int_E |f|^{p_1q} \right)^{1/q} \left( \int E 1^x \right)^{1/x} \leq \left( \int E |f|^{p_2} \right)^{p_1/p_2} |E|^{-\frac{p_2-p_1}{p_2}}.$$
Taking both sides to the power \(1/p_1\) yields:
\[
\|f\|_{p_1} = \left( \int_E |f|^{p_2} \right)^{1/p_2} |E|^{\frac{p_2-p_1}{p_1p_2}}
\]
But
\[
N_p[f] = \left( \frac{1}{|E|} \int_E |f|^p \right)^{1/p} = \frac{1}{|E|^{1/p}} \|f\|_p,
\]
so
\[
N_{p_1}[f] = \frac{1}{|E|^{1/p_1}} \|f\|_{p_1}
= \left( \int_E |f|^{p_2} \right)^{1/p_2} |E|^{-\frac{1}{p_2}}
= N_{p_2}[f].
\]

Now, by Minkowski,
\[
N_p[f + g] = \frac{1}{|E|^{1/p}} \|f + g\|_p
\leq \frac{1}{|E|^{1/p}} (\|f\|_p + \|g\|_p)
= N_p[f] + N_p[g].
\]

And, by Holder,
\[
\frac{1}{|E|} \int |fg| \leq \frac{1}{|E|^{1/p}} \|f\|_p \frac{1}{|E|^{1/p'}} \|g\|_{p'}
= N_p[f]N_{p'}[g].
\]

Finally, as \(|E|\) finite and non-zero,
\[
\lim_{p \to \infty} N_p[f] = \lim_{p \to \infty} \frac{1}{|E|^{1/p}} \|f\|_p = \lim_{p \to \infty} \|f\|_p = \|f\|_\infty.
\]
With this we have completed the problem.

**Exercise 0.8.** Chapter 8, # 6: Prove the generalization of Holder’s inequality. If \(\sum_{i=1}^k 1/p_i = 1/r, p_i, r \geq 1\), then
\[
\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.
\]

**Proof.** We proceed by induction considering first the case \(n = 2\).
\[
\|f_1 f_2\|_r = \int |f_1 f_2|^r \leq \|f_1|^r\|_{p_1/r} \|f_2|^r\|_{p_2/r}
= \left( \int |f_1|^{r(p_1/r)} \right)^{r/p_1} \left( \int |f_2|^{r(p_2/r)} \right)^{r/p_2}
= \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r.
\]
Here the first inequality was by Holder. Suppose now that this holds true for \( n - 1 \). For the case \( n \), we apply our result for \( n = 2 \) with now by replacing \( f_2 \) with \( \prod_{j=2}^{n} f_j \) and \( p_2 \) replaced by \( p' = \left( \sum_{j=2}^{n} 1/p_j \right)^{-1} \), then

\[
\| \prod_{j=1}^{n} f_j \| = \| f_1 \prod_{j=2}^{n} f_j \| \leq \| f_1 \|_{p_1} \| \prod_{j=2}^{n} f_j \|_{p'}.
\]

Now apply the \( n - 1 \) equality and we are finished. 

**Exercise 0.9.** Chapter 8, # 9: If \( f \) is real-valued and measurable on \( E \), define its essential infimum on \( E \) by

\[
\text{ess inf}_E f = \sup \{ \alpha : |\{ x \in E : f(x) < \alpha \}| = 0 \}.
\]

If \( f \leq 0 \), show that \( \text{ess inf}_E f = (\text{ess sup}_E 1/f)^{-1} \).

**Proof.** It is the greatest number \( \alpha \) such that \( f(x) \geq \alpha \) except for on a subset of measure zero.

\[
\text{ess inf}_E f = \sup \{ \alpha : |\{ x \in E : f(x) < \alpha \}| = 0 \} \\
= \inf \{ \alpha : |\{ x \in E : f(x) < \alpha^{-1} \}| = 0 \}^{-1} \\
= \inf \{ \alpha : |\{ x \in E : f(x)^{-1} > \alpha \}| = 0 \}^{-1} \\
= (\text{ess sup}_E 1/f)^{-1}.
\]