Problem Set 2 Due Wednesday February 24 in class

You may discuss problems with classmates but write your own solutions. Write carefully and rigorously and to the point, preferably using LaTeX.

Read sections 4.6, 2.2.

1. Solve the Cauchy problem

$$u_{xx} + u_{yy} = 0 \quad u(x,0) = x^2,\ u_y(x,0) = e^x \ , \tag{0.1}$$

by reducing to an equivalent first order system. Either by using this first order system or (even easier) working directly with (0.1), write down the Taylor series of the solution near (0,0) up to the power $x^\alpha y^\beta$, $\alpha + \beta \le 3$.

2. Suppose $p = \sum_{n=0}^\infty p_n x^n$, $q = \sum_{n=0}^\infty q_n x^n$, $f = \sum_{n=0}^\infty f_n x^n$, are given analytic functions with

$$|p_n| + |q_n| + |f_n| \le \frac{M}{\rho^n} \quad M,\ \rho > 0 \text{ are given constants} \tag{0.2}$$

Consider the ode problem

$$u'' = p(x)u' + q(x)u + f(x) \quad \text{in } |x| < \rho \quad u(0) = u_0,\ u'(0) = u_1 \tag{0.3}$$

where u_0, u_1 are given constants.

Show that if $u = \sum_{j=0}^\infty a_j x^j$ is an analytic solution of (0.3), there is a recurrence relation giving a_{n+1} in terms of

$$u_0,\ u_1,\ a_0, \dots a_n,\ q_0, \dots q_n,\ f_0. \dots, f_n \ .$$

Show that the series $\sum_{j=0}^\infty a_j x^j$ is majorized by the power series solution of

(0.4)

$$U'' = P(x)u' + Q(x)u + F \quad \text{in } |x| < \rho \quad U(0) = |u_0|,\ U'(0) = |u_1| \ ,$$

where $P = \frac{M\rho}{\rho - x}$, $Q = F = \frac{M\rho^2}{(\rho-x)^2}$. Hence prove (0.3) has an analytic solution in $|x| < \rho$.

3. Show that a bounded harmonic function in R^n is constant. Deduce the Fundamental Theorem of Algebra: A complex polynomial p(z) in the plane always has a root.

4. Show that a positive harmonic function in R^n is constant.

5. Show that if u is harmonic in R^n and $u = o(|x|)$, then u is constant. (Hint: Use the solid version of the mean value property $u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)dy$ and estimate $|u(x) - u(x')|$.)

6. Let u be harmonic in R^n with $|u(x)| \leq C(1 + |x|^N)$ for constants C,N. Show that u is a polynomial of degree at most N.