EMBEDDED MINIMAL TORI IN S^3 AND THE LAWSON CONJECTURE

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Abstract. We show that any embedded minimal torus in S^3 is congruent to the Clifford torus. This answers a question posed by H.B. Lawson, Jr., in 1970.

1. Introduction

The study of minimal surfaces is one of the oldest subjects in differential geometry. Of particular interest are minimal surfaces in spaces of constant curvature, such as the Euclidean space $\mathbb{R}^3$ or the sphere S^3. The case of the sphere S^3 turns out to be very interesting: for example, while there are no closed minimal surfaces in $\mathbb{R}^3$, the sphere S^3 does contain closed minimal surfaces. The simplest example of a minimal surface in S^3 is the equator. Another basic example is the so-called Clifford torus. Identifying S^3 with the unit sphere in $\mathbb{R}^4$, the Clifford torus is defined by

$$\Big\{(x_1, x_2, x_3, x_4) \in S^3 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}\Big\}.$$

We note that the principal curvatures of the Clifford torus are 1 and -1, and the intrinsic Gaussian curvature vanishes identically.

Lawson [15] proved that, given any positive integer g, there exists at least one compact embedded minimal surface in S^3 with genus g (cf. [15], Section 6). Moreover, it was shown in [15] that there are at least two such surfaces unless the genus g is a prime number. Additional examples of compact embedded minimal surfaces in S^3 were later found by Karcher, Pinkall, and Sterling [13] and by Kapouleas and Yang [12]. The construction of Karcher, Pinkall, and Sterling uses tesselations of S^3 into cells that have the symmetry of a Platonic solid in $\mathbb{R}^3$; the resulting minimal surfaces have genus 3, 5, 6, 7, 11, 19, 73, and 601, respectively. The result of Kapouleas and Yang relies on a doubling construction: roughly speaking, this construction involves joining together two copies of the Clifford torus by gluing in a large number of catenoid necks. The resulting surfaces have small mean curvature, and Kapouleas and Yang employed the implicit function theorem to deform these surfaces to exact solutions of the minimal surface equation. We note that

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Kapouleas has recently described a similar doubling construction involving the equator instead of the Clifford torus (cf. [11], Section 2.4).

Using the method of Hopf differentials, Almgren showed that any immersed minimal two-sphere in S^3 is totally geodesic, and therefore congruent to the equator (see [15], Proposition 1.5). In 1970, Lawson [16] conjectured that the Clifford torus is the only compact embedded minimal surface in S^3 of genus 1. In this paper, we give an affirmative answer to Lawson's conjecture:

Theorem 1. *Suppose that $F : \Sigma \to S^3$ is an embedded minimal torus in S^3. Then F is congruent to the Clifford torus.*

We note that the embeddedness assumption in Theorem 1 is crucial: in fact, Hsiang and Lawson [9] have constructed an infinite family of immersed minimal tori in S^3.

Lawson's conjecture has attracted considerable interest over the past decades, and various partial results are known. For example, it was shown by Urbano [21] that any minimal torus in S^3 which has Morse index at most 5 is congruent to the Clifford torus. Moreover, Ros [18] was able to verify Lawson's conjecture for surfaces that are invariant under reflection across each coordinate plane. Finally, Montiel and Ros [17] linked Lawson's conjecture to a conjecture of Yau (cf. [22]) concerning the first eigenvalue of the Laplacian on a minimal surface. Yau's conjecture is discussed in more detail in [5], [6], and [7].

Our method of proof is inspired in part by the pioneering work of G. Huisken [10] on the curve shortening flow, as well as by recent work of B. Andrews [1] on the mean curvature flow. Let us digress briefly to review these results.

Given a one-parameter family of embedded curves $F_t : S^1 \to \mathbb{R}^2$, Huisken considered the quantity

$$W_t(x,y) = \frac{L(t)}{|F_t(x) - F_t(y)|} \sin\Big(\frac{\pi\, d_t(x,y)}{L(t)}\Big),$$

where $L(t)$ denotes the total length of the curve F_t and $d_t(x,y)$ denotes the intrinsic distance of two points $x, y \in S^1$. Huisken discovered that if the curves F_t evolve by the curve shortening flow, then the supremum of the function $Z_t(x,y)$ is monotone decreasing in t. As a result, Huisken obtained a new and direct proof of a theorem of Grayson [8].

This technique was developed further in an important recent paper by B. Andrews [1]. In this paper, Andrews considered a one-parameter family of embedded hypersurfaces $F_t : M \to \mathbb{R}^{n+1}$ which have positive mean curvature and evolve by the mean curvature flow. By applying the maximum principle to a suitable function $W_t(x,y)$ defined on $M \times M$, Andrews obtained a direct proof of the noncollapsing estimate established earlier by Sheng and Wang [19] (see also [23]). We note that the argument in [1] relies in a crucial way on the positivity of the mean curvature; in particular, the argument does not seem to be applicable in the case of minimal surfaces.

We now describe the main ideas involved in the proof of Theorem 1. Let $F : \Sigma \to S^3$ be an embedded minimal torus in S^3. It follows from work of Lawson that the surface Σ has no umbilic points. Hence, if we choose κ sufficiently large, then the quantity

$$Z(x, y) = \frac{\kappa}{\sqrt{2}} \, |A(x)| \, (1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle$$

is nonnegative for all points $x, y \in \Sigma$. We now consider the smallest real number κ with the property that $Z(x, y) \geq 0$ for all points $x, y \in \Sigma$. It is easy to see that $\kappa \geq 1$. Using the maximum principle, we are able to show that $\kappa = 1$. This involves a very intricate calculation which exploits special identities arising from the first and second variations of the function $Z(x, y)$; see Section 2 for details. Once we have shown that $\kappa = 1$, we are able to show that the second fundamental form of F is parallel. From this, we deduce that the induced metric on Σ is flat. A classical theorem of Lawson [14] then implies that F is congruent to the Clifford torus.

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2. The key technical ingredient

Let $F : \Sigma \to S^3$ be an embedded minimal surface in S^3 (viewed as the unit sphere in $\mathbb{R}^4$). Moreover, let Φ be a positive function on Σ. We consider the expression

$$Z(x, y) = \Phi(x) \, (1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle.$$

Let us consider a pair of points $\bar{x} \neq \bar{y}$ with the property that $Z(\bar{x}, \bar{y}) = 0$ and the differential of Z at the point $(\bar{x}, \bar{y})$ vanishes. Let (x_1, x_2) be geodesic normal coordinates around $\bar{x}$, and let (y_1, y_2) be geodesic normal coordinates around $\bar{y}$.

At the point $(\bar{x}, \bar{y})$, we have

$$\begin{aligned} 0 = \frac{\partial Z}{\partial x_i}(\bar{x}, \bar{y}) &= \frac{\partial \Phi}{\partial x_i}(\bar{x}) \, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) \\ &\quad - \Phi(\bar{x}) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle + h_i^k(\bar{x}) \Big\langle \frac{\partial F}{\partial x_k}(\bar{x}), F(\bar{y}) \Big\rangle \end{aligned}$$

and

$$0 = \frac{\partial Z}{\partial y_i}(\bar{x}, \bar{y}) = -\Phi(\bar{x}) \Big\langle F(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle + \Big\langle \nu(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle.$$

We will make extensive use of these relations in the subsequent arguments.

Without loss of generality, we may assume that the second fundamental form at $\bar{x}$ is diagonal, so that $h_{11}(\bar{x}) = \lambda_1$, $h_{12}(\bar{x}) = 0$, and $h_{22}(\bar{x}) = \lambda_2$. We denote by w_i the reflection of the vector $\frac{\partial F}{\partial x_i}(\bar{x})$ across the hyperplane orthogonal to $F(\bar{x}) - F(\bar{y})$, i.e.

$$w_i = \frac{\partial F}{\partial x_i}(\bar{x}) - 2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), \frac{F(\bar{x}) - F(\bar{y})}{|F(\bar{x}) - F(\bar{y})|} \Big\rangle \frac{F(\bar{x}) - F(\bar{y})}{|F(\bar{x}) - F(\bar{y})|}.$$

By a suitable choice of the coordinate system (y_1, y_2), we can arrange that $\big\langle w_1, \frac{\partial F}{\partial y_1}(\bar{y}) \big\rangle \geq 0$, $\big\langle w_1, \frac{\partial F}{\partial y_2}(\bar{y}) \big\rangle = 0$, and $\big\langle w_2, \frac{\partial F}{\partial y_2}(\bar{y}) \big\rangle \geq 0$.

Lemma 2. *The vectors $F(\bar{y})$ and $\Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x})$ are linearly independent.*

Proof. Using the identity

$$\langle \Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x}), F(\bar{y}) \rangle = \Phi(\bar{x}),$$

we obtain

$$\begin{aligned} &|\Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x})|^2\, |F(\bar{y})|^2 - \langle \Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x}), F(\bar{y}) \rangle^2 \\ &= |\Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x})|^2 - \Phi(\bar{x})^2 = 1. \end{aligned}$$

From this, the assertion follows.

Lemma 3. *We have $w_1 = \frac{\partial F}{\partial y_1}(\bar{y})$ and $w_2 = \frac{\partial F}{\partial y_2}(\bar{y})$.*

Proof. A straightforward calculation gives

$$\begin{aligned} \langle w_i, F(\bar{y}) \rangle &= \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle \\ &\quad + 2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle\, \frac{\langle F(\bar{x}) - F(\bar{y}), F(\bar{y}) \rangle}{|F(\bar{x}) - F(\bar{y})|^2} = 0 \end{aligned}$$

and

$$\begin{aligned} &\langle w_i, \Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x}) \rangle \\ &= 2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle\, \frac{\langle F(\bar{x}) - F(\bar{y}), \Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x}) \rangle}{|F(\bar{x}) - F(\bar{y})|^2} \\ &= 2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle\, \frac{Z(\bar{x}, \bar{y})}{|F(\bar{x}) - F(\bar{y})|^2} = 0. \end{aligned}$$

On the other hand, the vectors $\frac{\partial F}{\partial y_1}(\bar{y})$ and $\frac{\partial F}{\partial y_2}(\bar{y})$ satisfy

$$\Big\langle \frac{\partial F}{\partial y_i}(\bar{y}), F(\bar{y}) \Big\rangle = 0$$

and

$$\Big\langle \frac{\partial F}{\partial y_i}(\bar{y}), \Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x}) \Big\rangle = -\frac{\partial Z}{\partial y_i}(\bar{x}, \bar{y}) = 0.$$

Since the vectors $F(\bar{y})$ and $\Phi(\bar{x})\, F(\bar{x}) - \nu(\bar{x})$ are linearly independent, we conclude that the plane spanned by w_1 and w_2 coincides with the plane spanned by $\frac{\partial F}{\partial y_1}(\bar{y})$ and $\frac{\partial F}{\partial y_2}(\bar{y})$. Moreover, w_1 and w_2 are orthonormal. Since $\big\langle w_1, \frac{\partial F}{\partial y_2}(\bar{y}) \big\rangle = 0$, we conclude that $w_1 = \pm \frac{\partial F}{\partial y_1}(\bar{y})$ and $w_2 = \pm \frac{\partial F}{\partial y_2}(\bar{y})$. Since $\big\langle w_1, \frac{\partial F}{\partial y_1}(\bar{y}) \big\rangle \geq 0$ and $\big\langle w_2, \frac{\partial F}{\partial y_2}(\bar{y}) \big\rangle \geq 0$, the assertion follows.

We next consider the second order derivatives of Z at the point $(\bar{x}, \bar{y})$.

Proposition 4. *We have*

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) \\
&= \Big(\Delta_\Sigma \Phi(\bar{x}) - \frac{|\nabla \Phi(\bar{x})|^2}{\Phi(\bar{x})} + (|A(\bar{x})|^2 - 2)\, \Phi(\bar{x})\Big)\, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) + 2\, \Phi(\bar{x}) \\
&- \frac{2\, \Phi(\bar{x})^2 - |A(\bar{x})|^2}{2\, \Phi(\bar{x})\, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle)} \sum_{i=1}^{2} \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle^2.
\end{aligned}$$

Proof. It follows from the Codazzi equations that

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} h_i^k(\bar{x}) = 0.$$

This implies

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) \\
&= \sum_{i=1}^{2} \frac{\partial^2 \Phi}{\partial x_i^2}(\bar{x})\, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) - 2 \sum_{i=1}^{2} \frac{\partial \Phi}{\partial x_i}(\bar{x}) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle \\
&+ 2\, \Phi(\bar{x})\, \langle F(\bar{x}), F(\bar{y}) \rangle - |A(\bar{x})|^2\, \langle \nu(\bar{x}), F(\bar{y}) \rangle \\
&= \big(\Delta_\Sigma \Phi(\bar{x}) + (|A(\bar{x})|^2 - 2)\, \Phi(\bar{x})\big)\, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) + 2\, \Phi(\bar{x}) \\
&- 2 \sum_{i=1}^{2} \frac{\partial \Phi}{\partial x_i}(\bar{x}) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle.
\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) \\
&= \Big(\Delta_\Sigma \Phi(\bar{x}) - \frac{|\nabla \Phi(\bar{x})|^2}{\Phi(\bar{x})} + (|A(\bar{x})|^2 - 2)\,\Phi(\bar{x})\Big)\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle) + 2\,\Phi(\bar{x}) \\
&+ \frac{1}{\Phi(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle)} \sum_{i=1}^{2} \Big(\frac{\partial \Phi}{\partial x_i}(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle) - \Phi(\bar{x}) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle\Big)^2 \\
&- \frac{\Phi(\bar{x})}{1 - \langle F(\bar{x}), F(\bar{y})\rangle} \sum_{i=1}^{2} \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle^2 \\
&= \Big(\Delta_\Sigma \Phi(\bar{x}) - \frac{|\nabla \Phi(\bar{x})|^2}{\Phi(\bar{x})} + (|A(\bar{x})|^2 - 2)\,\Phi(\bar{x})\Big)\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle) + 2\,\Phi(\bar{x}) \\
&+ \frac{1}{\Phi(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle)} \sum_{i=1}^{2} \lambda_i^2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle^2 \\
&- \frac{\Phi(\bar{x})}{1 - \langle F(\bar{x}), F(\bar{y})\rangle} \sum_{i=1}^{2} \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle^2.
\end{aligned}$$

Since $\lambda_1^2 = \lambda_2^2 = \frac{1}{2}\,|A(\bar{x})|^2$, the assertion follows.

Proposition 5. *We have*

$$\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) = \lambda_i - \Phi(\bar{x}).$$

Proof. Using Lemma 3, we obtain

$$\begin{aligned}
&\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) \\
&= -\frac{\partial \Phi}{\partial x_i}(\bar{x}) \Big\langle F(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle + (\lambda_i - \Phi(\bar{x})) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle \\
&= \frac{1}{1 - \langle F(\bar{x}), F(\bar{y}) \rangle} \, (\lambda_i - \Phi(\bar{x})) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle \Big\langle F(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle \\
&+ (\lambda_i - \Phi(\bar{x})) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle \\
&= -2\,(\lambda_i - \Phi(\bar{x})) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), \frac{F(\bar{x}) - F(\bar{y})}{|F(\bar{x}) - F(\bar{y})|} \Big\rangle \Big\langle \frac{F(\bar{x}) - F(\bar{y})}{|F(\bar{x}) - F(\bar{y})|}, \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle \\
&+ (\lambda_i - \Phi(\bar{x})) \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle \\
&= (\lambda_i - \Phi(\bar{x})) \Big\langle w_i, \frac{\partial F}{\partial y_i}(\bar{y}) \Big\rangle \\
&= \lambda_i - \Phi(\bar{x}),
\end{aligned}$$

as claimed.

Proposition 6. *We have*

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2 \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) \\
&= \Big(\Delta_\Sigma \Phi(\bar{x}) - \frac{|\nabla \Phi(\bar{x})|^2}{\Phi(\bar{x})} + (|A(\bar{x})|^2 - 2)\,\Phi(\bar{x})\Big)\,(1 - \langle F(\bar{x}), F(\bar{y}) \rangle) \\
&- \frac{2\,\Phi(\bar{x})^2 - |A(\bar{x})|^2}{2\,\Phi(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y}) \rangle)} \sum_{i=1}^{2} \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle^2.
\end{aligned}$$

Proof. By Proposition 5, we have

$$\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) = \sum_{i=1}^{2} (\lambda_i - \Phi(\bar{x})) = -2\,\Phi(\bar{x}).$$

Moreover, we have

$$\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) = 2\,\Phi(\bar{x})\,\langle F(\bar{x}), F(\bar{y}) \rangle - 2\,\langle \nu(\bar{x}), F(\bar{y}) \rangle = 2\,\Phi(\bar{x}).$$

Using these identities in combination with Proposition 4, we conclude that

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2 \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i \, \partial y_i}(\bar{x},\bar{y}) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) \\
&= \Big(\Delta_\Sigma \Phi(\bar{x}) - \frac{|\nabla \Phi(\bar{x})|^2}{\Phi(\bar{x})} + (|A(\bar{x})|^2 - 2) \, \Phi(\bar{x}) \Big) \, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) \\
&- \frac{2 \, \Phi(\bar{x})^2 - |A(\bar{x})|^2}{2 \, \Phi(\bar{x}) \, (1 - \langle F(\bar{x}), F(\bar{y}) \rangle)} \sum_{i=1}^{2} \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle^2.
\end{aligned}$$

This completes the proof.

3. Proof of Theorem 1

We now specify our choice of the function Φ. We will put $\Phi(x) = \kappa \, \Psi(x)$, where κ is a positive constant and $\Psi(x) = \frac{1}{\sqrt{2}} \, |A(x)|$.

Proposition 7. *Suppose that $F : \Sigma \to S^3$ is an embedded minimal torus in S^3. Then the function $\Psi = \frac{1}{\sqrt{2}} \, |A|$ is strictly positive. Moreover, Ψ satisfies the partial differential equation*

$$\Delta_\Sigma \Psi - \frac{|\nabla \Psi|^2}{\Psi} + (|A|^2 - 2) \, \Psi = 0.$$

Proof. It follows from work of Lawson that a minimal torus in S^3 has no umbilical points (see [15], Proposition 1.5). Thus, the function $|A|$ is strictly positive everywhere. Using the Simons identity (cf. [20], Theorem 5.3.1), we obtain

$$\Delta h_{ik} + (|A|^2 - 2) \, h_{ik} = 0,$$

hence

$$\Delta_\Sigma(|A|^2) - 2 \, |\nabla A|^2 + 2 \, (|A|^2 - 2) \, |A|^2 = 0.$$

The Codazzi equations imply that $|\nabla A|^2 = 2 \, \big|\nabla |A|\big|^2$. Consequently, we have

$$\Delta_\Sigma(|A|) - \frac{\big|\nabla |A|\big|^2}{|A|} + (|A|^2 - 2) \, |A| = 0,$$

as claimed.

Theorem 8. *Suppose that $F : \Sigma \to S^3$ is an embedded minimal torus in S^3. Then the induced metric on Σ is flat.*

Proof. For each point $x \in \Sigma$, we have

$$\limsup_{y \to x} \Big(- \frac{\langle \nu(x), F(y) \rangle}{\Psi(x) \, (1 - \langle F(x), F(y) \rangle)} \Big) = 1.$$

Let us define

$$\kappa = \sup_{x,y \in \Sigma, \, x \neq y} \Big(- \frac{\langle \nu(x), F(y) \rangle}{\Psi(x) \, (1 - \langle F(x), F(y) \rangle)} \Big)$$

and

$$Z(x,y) = \kappa\,\Psi(x)\,(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0.$$

Clearly, $\kappa \geq 1$ and $Z(x,y) \geq 0$. We now distinguish two cases:

Case 1: Suppose that $\kappa > 1$. In this case, the set

$$\Omega = \{\bar{x} \in \Sigma : \text{there exists a point } \bar{y} \in \Sigma \setminus \{\bar{x}\} \text{ such that } Z(\bar{x},\bar{y}) = 0\}$$

is non-empty. By Proposition 6, we have

$$\begin{aligned} &\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) + \sum_{i=1}^2 \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) \\ &= -\frac{\kappa^2 - 1}{\kappa}\,\frac{\Psi(\bar{x})}{1 - \langle F(\bar{x}), F(\bar{y})\rangle} \sum_{i=1}^2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle^2 \leq 0 \end{aligned}$$

for every pair of points $\bar{x} \neq \bar{y}$ with the property that $Z(\bar{x},\bar{y}) = \frac{\partial Z}{\partial x_i}(\bar{x},\bar{y}) = \frac{\partial Z}{\partial y_i}(\bar{x},\bar{y}) = 0$. Using the same arguments, we can show that

$$\begin{aligned} &\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(x,y) + \sum_{i=1}^2 \frac{\partial^2 Z}{\partial y_i^2}(x,y) \\ &\leq \Lambda(x,y)\,\bigg(Z(x,y) + \sum_{i=1}^2 \Big|\frac{\partial Z}{\partial x_i}(x,y)\Big| + \sum_{i=1}^2 \Big|\frac{\partial Z}{\partial y_i}(x,y)\Big|\bigg) \end{aligned}$$

for all points $x \neq y$. Here, $\Lambda(x,y)$ is a continuous function on the set $\{(x,y) \in \Sigma \times \Sigma : x \neq y\}$, which may approach infinity along the diagonal. Hence, Bony's strict maximum principle for degenerate elliptic equations implies that Ω is open (see [3] or [4], Corollary 9.7).

We now consider a point $\bar{x} \in \Omega$. By definition of Ω, we can find a point $\bar{y} \in \Sigma \setminus \bar{x}$ such that $Z(\bar{x},\bar{y}) = 0$. Since the function Z attains its global minimum at the point $(\bar{x},\bar{y})$, we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) + \sum_{i=1}^2 \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) \\ &= -\frac{\kappa^2 - 1}{\kappa}\,\frac{\Psi(\bar{x})}{1 - \langle F(\bar{x}), F(\bar{y})\rangle} \sum_{i=1}^2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle^2 \leq 0. \end{aligned}$$

Since $\kappa > 1$, we conclude that

$$\Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y})\Big\rangle = 0$$

for each i. From this, we deduce that

$$0 = \frac{\partial Z}{\partial x_i}(\bar{x},\bar{y}) = \kappa\,\frac{\partial \Psi}{\partial x_i}(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle)$$

for each i. Therefore, $\nabla \Psi(\bar{x}) = 0$ for each point $\bar{x} \in \Omega$. Since Ω is open, it follows that $\Delta_\Sigma \Psi(\bar{x}) = 0$ for each point $\bar{x} \in \Omega$. Hence, Proposition 7, implies that $\Psi(\bar{x}) = 1$ for each point $\bar{x} \in \Omega$. Using standard unique continuation theorems for solutions of elliptic partial differential equations (see e.g. [2]), we conclude that $\Psi(x) = 1$ for all $x \in \Sigma$. Consequently, the Gaussian curvature of Σ vanishes identically.

Case 2: We now consider the case $\kappa = 1$. In this case, we have

$$Z(x,y) = \Psi(x)\,(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0$$

for all points x, y. For simplicity, let us identify the surface Σ with its image under the embedding F, so that $F(x) = x$. Let us fix an arbitrary point $\bar{x} \in \Sigma$. We can find an orthonormal basis $\{e_1, e_2\}$ of $T_{\bar{x}}\Sigma$ such that $h(e_1, e_1) = \Psi(\bar{x})$, $h(e_1, e_2) = 0$, and $h(e_2, e_2) = -\Psi(\bar{x})$. Let $\gamma(t)$ be a geodesic on Σ such that $\gamma(0) = \bar{x}$ and $\gamma'(0) = e_1$. We define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = Z(\bar{x}, \gamma(t)) = \Psi(\bar{x})\,(1 - \langle \bar{x}, \gamma(t)\rangle) + \langle \nu(\bar{x}), \gamma(t)\rangle \geq 0.$$

A straightforward calculation gives

$$f'(t) = -\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \gamma'(t)\rangle,$$

$$\begin{aligned} f''(t) &= \langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \gamma(t)\rangle \\ &\quad + h(\gamma'(t), \gamma'(t))\,\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \nu(\gamma(t))\rangle, \end{aligned}$$

and

$$\begin{aligned} f'''(t) &= \langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \gamma'(t)\rangle \\ &\quad + h(\gamma'(t), \gamma'(t))\,\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), D_{\gamma'(t)}\nu\rangle \\ &\quad + (D^\Sigma_{\gamma'(t)} h)(\gamma'(t), \gamma'(t))\,\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \nu(\gamma(t))\rangle. \end{aligned}$$

In particular, we have $f(0) = f'(0) = f''(0) = 0$. Since $f(t)$ is nonnegative, we conclude that $f'''(0) = 0$. From this, we deduce that $(D^\Sigma_{e_1} h)(e_1, e_1) = 0$. An analogous argument with $\{e_1, e_2, \nu\}$ replaced by $\{e_2, e_1, -\nu\}$ yields $(D^\Sigma_{e_2} h)(e_2, e_2) = 0$. Using these identities and the Codazzi equations, we conclude that the second fundamental form is parallel. In particular, the intrinsic Gaussian curvature of Σ is constant. Consequently, the induced metric on Σ is flat. This completes the proof of Theorem 8.

We now complete the proof of Theorem 1. By Theorem 8, the induced metric on Σ is flat. On the other hand, Lawson [14] proved that the Clifford torus is the only flat minimal torus in S^3. Putting these facts together, the assertion follows.

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