On the Regularity of the Solution of the $n$-Dimensional Minkowski Problem

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0. Introduction

Given a closed strictly convex hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$, the Gauss map of $M$ defines a homeomorphism between $M$ and the unit sphere $S^n$. Therefore the Gauss–Kronecker curvature of $M$ can be transplanted via the Gauss map to a function defined on $S^n$. If this function is denoted by $K$, then Minkowski observed that $K$ must satisfy the equation

$$\int_{S^n} x_i K^{-1} = 0,$$

where $x_i$ are the coordinate functions on $S^n$.

Minkowski then asked the converse of the problem. Namely, given a positive function $K$ defined on $S^n$ satisfying the above integral conditions, can we find a closed strictly convex hypersurface whose curvature function is given by $K$? Minkowski solved the problem in the category of polyhedrons. Then A. D. Alexandrov and others solved the problem in general. However, this last solution does not provide any information about the regularity of the (unique) convex hypersurface even if we assume $K$ is smooth.

In the two-dimensional case, H. Lewy was the first one who proved that if $K$ is analytic, the solution to the Minkowski problem is also analytic. Around 1953, A. V. Pogorelov [9] and L. Nirenberg [6] solved the regularity problem in the smooth category independently. Their methods were quite different and restricted only to two dimensions. The method of Pogorelov was to show that the (unique) generalized solution of Alexandrov is smooth. This depends on the solvability and regularity of the Dirichlet problem of the two-dimensional Monge–Ampère equation. The method of L. Nirenberg was to use the continuity method to produce a smooth solution directly. This depends on a priori estimates which are available only for elliptic equations of two variables.

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For dimensions higher than two, the regularity problem was unknown until A. V. Pogorelov announced the solution in [8]. He derived in this paper an estimate of the derivatives of third order using a computation of E. Calabi. Then, without further proof, he claimed the rest of the proof is the same as the one in two dimensions. However, as was indicated above, his proof in two dimensions depends on the regularity of the Monge–Ampère equation. In higher dimensions, this was not known at that time. Two years later, he announced in [7] the regularity of the higher-dimensional Monge–Ampère equation. Unfortunately, besides the gaps in his proof (see the comments in [4]) the proof of the regularity depends on the solution of the regularity of the Minkowski problem.

The purpose of this paper is to provide a proof of the regularity of the higher-dimensional Minkowski problem without assuming the regularity of the Monge–Ampère equation. Our method is the continuity method which was used by L. Nirenberg in [6]. However one notices that even if we assume the a priori estimates of the third derivatives of the Monge–Ampère equation, Nirenberg's method cannot be generalized immediately. For example, Bonnet's theorem tells us that for a complete Riemannian manifold with Ricci curvature bounded from below by a positive constant, the diameter can be estimated from above depending only on that constant. This theorem is readily applicable to the two-dimensional Minkowski problem but not to the higher-dimensional problem. Furthermore, since we have only interior estimates for the third derivatives, we need Lemma 4 which is not easy. The strategy of studying differential equations on the sphere was also used by L. Nirenberg. But the generalization is not trivial.

For the purpose of proving the regularity of the Monge–Ampère equation in [4], we briefly recall some known facts about the generalized Minkowski problem. The important fact that we need in [4] is that when a sequence of area functions of convex bodies converges to an area function of a convex body in the $L^1$-sense, then a subsequence of the support functions of the convex bodies converge uniformly to the support function of the corresponding convex body.

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1. A Priori Estimates

In this section, we record the estimates obtained by Pogorelov [7], [8] where, in the case of the third-order estimates, an essential idea of Calabi [3] was used.

**Lemma 1.** Let $\Omega$ be an $n$-dimensional convex domain and $F \in C^2(\Omega)$ with $F > 0$. Let $u \in C^4(\Omega) \cap C^{0,1}(\Omega)$ be a strictly convex solution of the equation $\det(u_{ij}) = F(x)$ on $\Omega$ and assume that $u|_{\partial \Omega} = a$. Then there is a constant $\alpha$
depending only on $n$, $\max_{\Omega} |\nabla u|$, $\max_{\Omega} |u-a|$ and the $C^{1,1}$-norm of $F$ on $\bar{\Omega}$ such that $\max_{x} |u_{ii}(x)| \leq \alpha |u-a|^{-1}$.

Proof: For any fixed direction $\partial/\partial x_{i}$, Pogorelov [7] considered the function $w = (a-u) e^{u_{ii}u_{ii}}$ on the domain $\Omega$. Then one may assume that $w$ attains its maximum at some point in $\bar{\Omega}$. (If $u_{ii}$ were not bounded in $\bar{\Omega}$, we could consider the function $(a-\varepsilon-u) e^{u_{ii}u_{ii}}$ in the domain $\{x \mid u(x) \leq a-\varepsilon\}$ and let $\varepsilon \to 0$.)

With a tricky calculation, Pogorelov derived the following inequality:

\begin{equation}
-\frac{nu_{ii}}{a-u} - \frac{u_{ii}^{2}}{(a-u)^{2}} + u_{ii}^{2} + u_{ii}F^{-1}F_{i} + F^{-1}F_{ii} - F^{2}F_{i}^{2} \leq 0.
\end{equation}

(1.1)

Multiplying (1.1) by $(a-u)^{2} e^{u_{ii}}$, one arrives at the inequality

\begin{equation}
w^{2} + Aw + B \leq 0,
\end{equation}

(1.2)

where $A$ and $B$ admit estimates depending on the quantities mentioned in the lemma.

The conclusion of the lemma follows readily from (1.2).

If the function $F(x)$ is bounded away from zero, then it is clear that an upper estimate of the quantities $u_{ii}$ also provides a lower estimate of $u_{ii}$. Hence if we assume that $F$ is positive in $\Omega$, then for a precompact subdomain of $\Omega$ we have upper and lower estimates of $u_{ii}$.

In order to have an interior estimate of the third derivative of $u$, it suffices therefore to estimate the quantity $\sum_{i,j,k,\alpha,\beta,\gamma} u_{i\alpha} u_{j\beta} u_{k\gamma} u_{ijk} u_{\alpha\beta\gamma}$. If one gives the graph of $u$ the affine metric defined in [4], the last quantity is dominated by the affine length of the Fubini–Pick form. In an intrinsic manner, this length was estimated in [4]. However, previous to [4], Pogorelov had already obtained the third-order estimates in a somewhat different manner. The essential ideas of all these approaches were due to E. Calabi [3].

\textbf{Lemma 2.} Let $\Omega$ be an $n$-dimensional convex domain and $F \in C^{3}(\bar{\Omega})$ with $F > 0$. Let $u \in C^{3}(\Omega) \cap C^{1,1}(\bar{\Omega})$ be a strictly convex solution to the equation $\det (u_{ii}) = F(x)$ and assume that $u|_{\partial \Omega} = a$. Then the third derivative of $u$ can be estimated by the $C^{1,1}$-norm of $u$, the $C^{2,1}$-norm of $F$, $\max_{x \in \Omega} 1/F(x)$ and the distance from $\partial \Omega$.

\textbf{2. The Setting of the $n$-Dimensional Minkowski Problem}

Let $K$ be a positive $C^{k,\alpha}$-function defined on the unit sphere $S^{n}$. Here $k \geq 3$ and $1 > \alpha > 0$. Suppose, for all coordinate functions $x_{i}$ on $S^{n}$, we have

\begin{equation}
\int_{S^{n}} x_{i}K^{-1} = 0.
\end{equation}

(2.1)
Then the Minkowski problem is to find a convex hypersurface $M$ in $\mathbb{R}^{n+1}$ such that under the identification of the Gauss map, the Gauss–Kronecker curvature of $M$ is given by $K$.

Given a strictly convex hypersurface $M$, we can define its support function $H$ as follows: Let $x = (x_1, \cdots, x_{n+1}) \in \mathbb{S}^n$ be any unit vector. Then we can find a (unique) point $y = (y_1, \cdots, y_{n+1}) \in M$ so that the Gauss image of $y$ is $x$. By definition we have $H(x) = \sum x_i y_i$. Extending this definition to $\mathbb{R}^{n+1}\{0\}$, we obtain a homogeneous function of degree one by the equation $H(x) = |x| H(x/|x|)$.

By the strict convexity of $M$, the coordinates $y_1, \cdots, y_{n+1}$ of $M$ may be considered as functions of $x_1, \cdots, x_{n+1}$ with $\sum_{i=1}^{n+1} x_i^2 = 1$. We extend these functions to be defined for all values of $(x_1, \cdots, x_{n+1})$ by requiring them to be homogeneous of degree zero. From the definition of $H$, it follows that

$$H(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} x_i y_i.$$  

(2.2)

We can therefore recover the coordinate functions of $M$ by the equations

$$y_i = \frac{\partial H}{\partial x_i}$$  

for $i = 1, \cdots, n+1$.

Since $y_i$ are functions of homogeneity of degree zero, they are completely determined by their values on the hyperplanes $x_i = -1$ for $i = 1, \cdots, n+1$.

As a typical example, we shall consider the hyperplane $x_{n+1} = -1$. Since $H$ can also be defined by

$$H(x) = \sup_{y \in M} \left( \sum_{i=1}^{n+1} x_i y_i \right),$$

$H$ is clearly a convex function. We shall restrict $H$ to the hyperplane $x_{n+1} = -1$ and compute its Hessian.

First of all, we note the following general fact. Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame of the sphere $\mathbb{S}^n$. Then restricting $H$ to $\mathbb{S}^n$ and taking its Hessian $(H_{ij})$ with respect to this frame, we have

$$\left(1 + \sum_{i=1}^{n} x_i^2\right)^{n/2+1} \det \left( \frac{\partial^2 H}{\partial x_i \partial x_j} \right)(x_1, \cdots, x_n, -1)$$

$$= \det (H_{ij} + H \delta_{ij}) \left( \frac{x_1}{\left(1 + \sum_{i=1}^{n} x_i^2\right)^{1/2}}, \cdots, \frac{x_n}{\left(1 + \sum_{i=1}^{n} x_i^2\right)^{1/2}}, \frac{-1}{\left(1 + \sum_{i=1}^{n} x_i^2\right)^{1/2}} \right).$$

(2.4)
Since (2.4) is a straightforward computation, using the homogeneity of $H$, we omit its calculation. On the other hand, it is a familiar fact (cf. [6]) that the right-hand side of (2.4) is given by $1/K$, where $K$ is the Gauss–Kronecker curvature of $M$ considered as a function on $S^n$.

Hence extending $K$ to be a function of homogeneity of degree zero, we have

$$\left(1 + \sum_{i=1}^{n} x_i^2\right)^{n/2+1} \det \left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right)(x_1, \cdots, x_n, -1) = \left(K(x_1, \cdots, x_n, -1)\right)^{-1}. \tag{2.5}$$

Obviously, equations similar to (2.5) should hold over the other hyperplanes $x_i = -1, i \leq n$. All these equations can combine into the following one on the unit sphere:

$$\det \left(H_{ij} + H \delta_{ij}\right) = \frac{1}{K}. \tag{2.6}$$

It follows easily from (2.5) that $H$ is strictly convex on the hyperplanes $x_i = -1$.

Conversely, suppose we can find a convex function of homogeneity of degree one satisfying equations similar to (2.5) (or equivalently (2.6)), then we can find a strictly convex hypersurface whose support function is $H$ and whose curvature function is $K$. Explicitly the coordinate functions of this hypersurface are obtained by the formulae (2.3) so that the smoothness of the hypersurface follows from the smoothness of $H$.

To see this assertion, we note that (2.3) defines functions with homogeneity of degree zero so that it is sufficient to consider their effects after restricting them to the hyperplanes $x_i = -1$. For simplicity we consider the strictly convex function $u(x_1, \cdots, x_n) = H(x_1, \cdots, x_n, -1)$.

Let $\Omega^* = \{(u_1(x), \cdots, u_n(x)) \mid x \in \mathbb{R}^n\}$. Then the Legendre transform of $u$ is the strictly convex function $u^*$ defined on $\Omega^*$ by the equation

$$u^* (u_1(x), \cdots, u_n(x)) = \sum_{i=1}^{n} x_i u_i(x) - u(x). \tag{2.7}$$

The normal of the graph of $u^*$ is given by

$$\left(1 + \sum_{i=1}^{n} \left(\frac{\partial u^*}{\partial y_i}\right)^2\right)^{-1/2} \left(\frac{\partial u^*}{\partial y_1}, \cdots, \frac{\partial u^*}{\partial y_n}, -1\right) \tag{2.8}$$

$$= \left(1 + \sum_{i=1}^{n} x_i^2\right)^{1/2} (x_1, \cdots, x_n, -1).$$
The support function of the graph of the function $u^*$ at the point 
\[(1 + \sum_{i=1}^{n} x_i^2)^{-1/2} (x_1, \cdots, x_n, -1)\]
is therefore equal to
\[
\left(1 + \sum_{i=1}^{n} x_i^2\right)^{-1/2} \left[\sum x_i u_i(x) - u^*(u_1(x), \cdots, u_n(x))\right] = \left(1 + \sum_{i=1}^{n} x_i^2\right)^{-1/2} u(x).
\]

Therefore, its value at $(x_1, \cdots, x_n, -1)$ is exactly $H(x_1, \cdots, x_n, -1)$. It remains to observe that
\[
\left(\frac{\partial H(x)}{\partial x_1}, \cdots, \frac{\partial H(x)}{\partial x_{n+1}}\right) = (u_1(x), \cdots, u_n(x), u^*(u_1(x), \cdots, u_n(x))).
\]

This follows immediately from the homogeneity of degree one of $H$.

In conclusion, the problem of Minkowski is reduced to the solution of the equation (2.6) on the unit sphere $S^n$ under the condition that $K$ is positive and satisfies the equation (2.1).

To solve this problem, we use the continuity method (cf. [6]). Thus let
\[
(2.9) \quad \frac{1}{K_t} = t/K + (1-t)
\]
for $0 \leq t \leq 1$. Then clearly all the functions $K_t$ are positive and satisfy the equation (2.1).

Let $S_\alpha = \{t \in [0, 1] | \text{the equation } \det (H_{ij} + H \delta_{ij}) = 1/K_t \text{ has a } C^{5,2,\alpha} \text{-solution } H \text{ on the sphere } S^n \text{ such that } (H_{ij} + H \delta_{ij}) > 0\}$. The method of continuity is divided into two steps. The first is to prove that $S_\alpha$ is closed in $[0, 1]$ and the second is to prove that $S_\alpha$ is open in $[0, 1]$. Since $0 \in S_\alpha$, this clearly provides $S_\alpha = [0, 1]$ and the equation (2.6) is solvable. The first step will be carried out in Section 3 and the second step will be carried out in Section 4.

3. The Closedness of the Set $S_\alpha$

Let $\{\delta_t\}$ be a sequence in $S$ which converges to a point $\delta_0 \in [0, 1]$. Let $H^l$ be a sequence of convex functions of homogeneity of degree one so that, restricting to $S^n$, $H^l$ are $C^{k+2,\alpha}$-functions with $(H^l_{ij} + H^l \delta_{ij}) > 0$ and satisfying the equations $\det (H^l_{ij} + H^l \delta_{ij}) = 1/K_{\alpha}$.

From the discussions in Section 2, we know that equations $y^l_1 = \partial H^l/\partial x_1$ define convex hypersurfaces $M^l$ whose support functions are given by $H^l$, respectively. We claim that the outer diameters of $M^l$ are uniformly bounded (independent of $l$).

Since we shall need it in the next paper, we state a lemma in a more precise form.
Lemma 3. Let $M$ be a compact convex $C^4$-hypersurface in $\mathbb{R}^{n+1}$. Let $K$ be its Gauss-Kronecker curvature function defined on $S^n$. Then the extrinsic diameter $L$ of $M$ can be estimated from above by the quantity

$$c_n \left( \int_{S^n} \frac{1}{K} \right)^{n/(n-1)} \left[ \inf_{u \in S^n} \int_{S^n} \max (0, \langle u, w \rangle) K(w)^{-1} \right]^{-1},$$

where $c_n$ is a positive constant depending only on $n$.

Proof: The lemma is known in the nonsmooth case (see [2]). (The assumption that $M$ is a $C^4$-hypersurface is made to insure that $K$ is defined. Of course we can replace this by the concept of generalized curvature.)

Let $L$ be the extrinsic diameter of $M$. Then there are two points $\{p, q\}$ on $M$ such that the line segment joining $p$ and $q$ has length $L$. Without loss of generality, we may assign our origin to be the midpoint of this segment. Let $u$ be a unit vector in the direction of this segment. Then for any unit vector $w \in S^n$, the support function of $M$ at $w$ is given by $H(w) = \sup_{y \in M} \langle y, w \rangle \geq \frac{L}{2} \max (0, \langle u, w \rangle)$.

Integrating over the sphere $S^n$, we have

$$L \leq 2 \left( \int_{S^n} \frac{H(w)}{K} \right) \left( \int_{S^n} \max (0, \langle u, w \rangle) K(w)^{-1} \right)^{-1}.$$  

(3.1)

The volume element of $M$ is $1/K$ times the volume element of $S^n$. Hence

$$\int_{S^n} \frac{H(w)}{K} = \int_{M} H.$$  

(3.2)

To compute $\int_{M} H$, we note that in $\mathbb{R}^{n+1}$

$$\Delta \left( \sum_{i=1}^{n+1} x_i^2 \right) = 2(n + 1).$$  

(3.3)

Applying the divergence formula to (3.3), it yields

$$\text{Vol} (\bar{M}) = \frac{1}{n+1} \int_{M} H,$$

(3.4)

where $\bar{M}$ is the volume of the body bounded by $M$.

The surface element of $M$ is given by $\int_{S^n} 1/K$. Hence the lemma is a consequence of (3.1), (3.2), (3.4) and the isoperimetric inequality.
It follows easily from Lemma 3 that the outer diameters of the convex hypersurfaces $M'$ are uniformly bounded from above. We are going to prove that the inner diameters of $M'$ are also bounded from below by a positive constant.

**Lemma 4.** In Lemma 3, we can find always a positive constant $r$ which depends only on an upper estimate of $\int_{S^n} 1/K$ and a lower estimate of $\inf_{u \in S^n} \int_{S^n} \max (0, \langle u, w \rangle) K(w)^{-1}$ so that we can always put a ball of radius $r$ inside the hypersurface $M$.

**Proof:** First of all, we find a lower bound of the areas of the projections of the surfaces on all possible hyperplanes.

Suppose we project the hypersurface $M$ along a unit vector $u$ onto a hyperplane perpendicular to $u$. Then clearly the volume of the body bounded by $M$ is not greater than the area of the projection multiplied by the outer diameter of $M$. By (3.4), (3.2) and (3.1), we conclude that the area of the projection is not less than

$$\frac{1}{2(n+1)} \inf_{u \in S^n} \left( \int_{S^n} \max (0, \langle u, w \rangle) K(w)^{-1} \right).$$

Since the volume of the body bounded by $M$ is dominated by $L^n$, (3.1) gives us a lower estimate of $L$. Hence we can find two points $p_1$ and $p_2$ on $M$ whose distance can be estimated from below. Consider the projection of $M$ on a hyperplane containing the segment $p_1p_2$. Since the outer diameter is bounded and the projected area is bounded from below, we can find a point $\tilde{p}_3$ on this hyperplane such that all the angles of the triangle $p_1p_2\tilde{p}_3$ lie between $\varepsilon$ and $\pi - \varepsilon$, where $\varepsilon$ is a positive constant estimated from below. Lifting $\tilde{p}_3$ to a point $p_3$ in $M$, we obtain a triangle $p_1p_2p_3$ with similar properties.

If $n > 2$, then we project $M$ into a hyperplane containing the triangle $p_1p_2p_3$. Arguing as above, we can find a point $p_4$ in $M$ such that the angles between all line segments issuing from $p_4$ to the $p_i$, $1 \leq i \leq 3$, and the plane of the triangle $p_1p_2p_3$ lie between $\varepsilon$ and $\pi - \varepsilon$, where $\varepsilon$ is a positive constant estimated from below. Continuing in this way, we obtain $n$ points $p_1, \ldots, p_n$ in $M$ such that the angles between all line segments issuing from $p_i$ to the $p_j$, $1 \leq j \leq i - 1$, in the plane of the simplex $p_1p_2 \cdots p_{i-1}$ lie between $\varepsilon$ and $\pi - \varepsilon$, where $\varepsilon$ is a positive constant estimated from below.

Finally, we project the hypersurface $M$ along a direction which is perpendicular to the hyperplane of the simplex $p_1p_2 \cdots p_n$. Then arguing as before, we can find a simplex $\tilde{p}_1\tilde{p}_2 \cdots \tilde{p}_{n+1}$ with properties similar to those of
The simplex \( \tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_{n+1} \) must lie in the body bounded by \( M \) and the properties ensure us of having a ball in its interior whose radius can be estimated from below. This finishes the proof of Lemma 4.

Putting both Lemma 3 and Lemma 4 together, we see that we can translate the \( M^t \) so that the origin lies in the interior of all the \( M_t \) and the numbers \( \inf_{x \in M} \|x\| \) and \( \sup_{x \in M} \|x\| \) are positive (finite) numbers. Corresponding to these translations, we change the support functions \( H^t \) by linear functions so that \( \inf_{x \in S^t} H^t(x) > 0 \) and \( \sup_{x \in S^t} H^t(x) < \infty \).

We are now in a position to apply Lemmas 1 and 2 to prove the closedness of \( S^t \).

Recall that, on a convex domain \( \Omega \), any convex function \( u \) defined on \( \Omega \) admits the estimate

\[
|\nabla u(x)| \leq \left( \sup_{\partial \Omega} u - u(x) \right) d(x, \partial \Omega)^{-1}.
\]

It follows from convexity of \( H^t \), from the fact that \( \sup_{x \in S^t} H^t(x) < \infty \), and from the homogeneity of \( H^t \) that we have a uniform gradient estimate of the \( H^t \) on each compact set of each hyperplane \( x_i = -1 \).

Since

\[
H^t(x_1, \cdots, x_n, -1) \geq \left( \inf_{x \in S^n} H^t(x) \right) \sqrt{1 + \sum_{i=1}^{n} x_i^2},
\]

one can see easily that for any positive number \( c \), the sets \( \Omega_l(c) = \{(x_1, \cdots, x_n, -1) \mid H^t(x_1, \cdots, x_n, -1) < c\} \) lie in a fixed compact set.

For any point \((a_1, \cdots, a_n, -1)\), we can find a positive number \( c \) depending only on \( \sup_{x \in S^t} H^t(x) \) and \( \sum_{i=1}^{n} a_i^2 \) such that \((a_1, \cdots, a_n, -1) \in \Omega_l(\frac{c}{2})\) for all \( l \).

Since all the sets \( \Omega_l(c) \) lie in a fixed compact set, the gradients of all the convex functions \( H^t \) are uniformly bounded and we can apply Lemmas 1 and 2 to the equations

\[
\left( 1 + \sum_{i=1}^{n} x_i^2 \right)^{n/2+1} \det \left( \frac{\partial^2 H^t}{\partial x_i \partial x_j} \right)(x_1, \cdots, x_n, -1) = (K_h(x_1, \cdots, x_n, -1))^{-1}
\]

to obtain uniform estimates of the second and the third derivatives of \( H^t \). (Note that we clearly have \( C^{2,1} \)-estimates of the functions \( K_h \).

A subsequence of the functions \( \{H^t\} \) therefore converges to a \( C^{2,1} \)-function \( H^0 \) which satisfies the following equation:

\[
\left( 1 + \sum_{i=1}^{n} x_i^2 \right)^{n/2+1} \det \left( \frac{\partial^2 H^0}{\partial x_i \partial x_j} \right)(x_1, \cdots, x_n, -1) = (K_h(x_1, \cdots, x_n, -1))^{-1}.
\]
Linearizing this equation and using the standard Schauder estimates, one can then prove that $H^0$ is a $C^{k+2,\alpha}$-function and $t_0 \in S_\alpha$. This finishes the proof of the closedness of $S_\alpha$.

4. The Openness of $S_\alpha$

In this case, we shall apply the implicit function theorem to the equation $\deg (H_{ij} + H \delta_{ij}) = K^{-1}$. Here $H$ is considered to be a function defined on the unit sphere.

Let $L_H$ be the linearized operator of the operator $H \to \det (H_{ij} + H \delta_{ij})$. Then for all smooth functions $u$ defined on $S^n$ we have

$$L_H(u) = \sum_{i,j} c(H_{ij} + H \delta_{ij})(u_{ij} + u \delta_{ij}),$$

where $c(H_{ij} + H \delta_{ij})$ are the cofactors of the matrix $(H_{ij} + H \delta_{ij})$.

We claim that, when $u$ and $v$ are functions in $C^2(S^n)$,

$$\int_{S^n} uL_H(v) = \int_{S^n} vL_H(u).$$

It is rather easy to see that the equation (4.2) is equivalent to the equation

$$\sum_i c(H_{ij} + H \delta_{ij}) = 0$$

for all $i$. (Differentiations here are covariant differentiations on $S^n$.)

We shall prove (4.3) under the assumption that $\det (H_{ij} + H \delta_{ij}) \neq 0$ in a neighborhood of the point which we are considering. The general case follows from this by approximating $H$ (in a neighborhood of this point).

Clearly, we have

$$\det (H_{pq} + H \delta_{pq})c(H_{ij} + H \delta_{ij}) = [\det (H_{pq} + H \delta_{pq})]c(H_{ij} + H \delta_{ij})$$

$$- c(H_{ip} + H \delta_{ip})(H_{pk} + H \delta_{pk})c(H_{kj} + H \delta_{kj}).$$

From (4.4), we know that

$$\det (H_{pq} + H \delta_{pq}) \sum_i c(H_{ij} + H \delta_{ij})$$

$$= \sum_{i,j,p} c(H_{ip} + H \delta_{ip})c(H_{kj} + H \delta_{kj})[H_{ijk} + H_{pj} \delta_{jk} - H_{pk} \delta_{ij}].$$
It remains to show that the terms in the bracket are zero. These are the commutation formulas and we derive them as follows:

Let \( \{\omega_i\} \) be a local orthonormal frame field and \( \{\omega_{ij}\} \) the corresponding connection forms. Then, by definition,

\[
\begin{align*}
(4.6) & \quad dH = \sum_i H_i \omega_i , \\
(4.7) & \quad \sum_j H_{ij} \omega_j = dH_i + \sum_j H_j \omega_{ji} , \\
(4.8) & \quad \sum_k H_{ijk} \omega_k = dH_{ij} + \sum_k H_{kj} \omega_{ki} + \sum_k H_{ik} \omega_{kj} .
\end{align*}
\]

Exterior differentiating (4.7), we obtain

\[
(4.9) \quad \sum_{j,k} H_{ijk} \omega_k \wedge \omega_j = -\frac{1}{2} \sum_{i,j,k} H_{ij} R_{lijk} \omega_k \wedge \omega_j ,
\]

where \( R_{lijk} \) is the curvature tensor of the sphere. Hence

\[
H_{ijk} - H_{ikj} = -\sum_i H_i R_{lijk} \\
= -H_k \delta_{ij} + H_j \delta_{ik} .
\]

By exterior differentiating (4.6), one finds that \( (H_{ij}) \) is symmetric and so

\[
H_{jk} - H_{pj} = (H_{jk} - H_{pj}) + (H_{pj} - H_{pk}) \\
= -H_p \delta_{jk} + H_k \delta_{jp} - H_k \delta_{pj} + H_j \delta_{pk} \\
= -H_p \delta_{jk} + H_j \delta_{pk} ,
\]

and our assertion (4.2) is proved.

We claim now, for any \( C^{5,\alpha} \)-function \( u \) defined on the sphere and for any coordinate function \( x_i \), that

\[
(4.12) \quad \int_{S^n} x_i \det (u_{pq} + u \delta_{pq}) = 0 .
\]

In fact

\[
L_i (u) = \int_{S^n} x_i \det (u_{pq} + u \delta_{pq})
\]
defines a differentiable functional on the Banach space of $C^{5,\alpha}$-functions defined on $S^n$. The Frechét derivative of this functional at a function $v$ is given by $\int_{S^n} x_i L_u(v)$. Since $L_u(x_i) = 0$, (4.2) shows that the Frechét derivative of $L_i$ is identically zero. Since $L_i(x_i) = 0$, $L_i$ is zero and (4.12) is proved.

**Remark.** In applications, $u$ is the support function of some strictly convex body and (4.12) can be proved more directly.

We are now in a position to prove the openness of $S_\alpha$. Let $B_1$ be the Banach space of $C^{k+2,\alpha}$-functions defined on $S^n$ and $B_2$ the Banach space of $C^{k,\alpha}$-functions $f$ defined on $S^n$ such that, for all coordinate functions $x_i$, $\int_{S^n} x_i f = 0$. Then according to (4.12), we can define a transformation $F : B_1 \rightarrow B_2$ by

\[
F(u) = \det (u_{ij} + u \delta_{ij}).
\]

We claim that if $H$ is the support function of some strictly convex hypersurface such that $H > 0$ on $S^n$, then for functions $f$ in a neighborhood of $\det (H_{ij} + H \delta_{ij})$ (in the topology of $B_2$) we can always solve the equation $F(u) = f$, where $u$ is also the support function of some strictly convex hypersurface.

As is well known (cf. [10]), it suffices to verify that the linearized operator of $F$ at $H$ is surjective. That is to say, the linear operator $L_H(u) = \sum_{i,j} c(H_{ij} + H \delta_{ij})(u_{ij} + u \delta_{ij})$ is surjective.

First of all, we find the kernel of $L_H$.

**Lemma 5.** Let $u$ be a function in $C^2(S^n)$ such that $L_H(u) = 0$, where $(H_{ij} + H \delta_{ij}) > 0$. Then, for some constants $a_1, \cdots, a_{n+1}$, $u = \sum_{i=1}^{n+1} a_i x_i$.

**Proof:** This lemma is closely related to the infinitesimal rigidity of the strictly convex body determined by $H$.

Consider the vector-valued function $Z = \sum_{i=1}^{n} u_i e_i + u e_{n+1}$, where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field and $e_{n+1}$ is the unit outer normal of $S^n$ (the subscript of $u$ means the differentiation with respect to the frame $\{e_1, \cdots, e_n\}$). Then $u = \langle Z, e_{n+1} \rangle$ and

\[
dZ = \sum_{j=1}^{n+1} \left( \sum_{i=1}^{n} (u_{ij} + u \delta_{ij}) e_i \right) \omega_j.
\]
Let \( X = \sum_{i=1}^{n} H_i e_i + H e_{n+1} \). Then as in (4.14), we have
\[
(4.15) \quad dX = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} (H_{ij} + H \delta_{ij}) e_i \right) \omega_j.
\]

Consider the \((n-1)\)-form \( \Omega = X \wedge Z \wedge dZ \wedge dX \wedge \cdots \wedge dX \), where \( dX \) appears \((n-2)\)-times. By computation,
\[
(4.16) \quad d\Omega = dX \wedge Z \wedge dZ \wedge dX \wedge \cdots \wedge dX + X \wedge dZ \wedge dX \wedge dX \wedge \cdots \wedge dX.
\]

The first term on the right-hand side of (4.16) is zero because \( dZ \wedge dX \wedge \cdots \wedge dX \), where \( dX \) appears \((n-1)\)-times, is
\[
\left[ \sum_{i,j} c(H_{ij} + H \delta_{ij})(u_{ij} + u \delta_{ij}) \right] (e_1 \wedge \cdots \wedge e_n) \otimes \omega_1 \wedge \cdots \wedge \omega_n.
\]

Integrating (4.16) over \( S^n \), we then obtain
\[
(4.17) \quad \int_{S^n} X \wedge dZ \wedge dZ \wedge dX \wedge \cdots \wedge dX = 0.
\]

Let \( \tilde{e}_j = \sum_i (H_{ij} + H \delta_{ij}) e_i \). Then
\[
(4.18) \quad dZ = \det (H_{ij} + H \delta_{ij})^{-1} \sum_{k,l,j} (u_{ij} + u \delta_{ij}) c(H_{kl} + H \delta_{kl}) \tilde{e}_k \otimes \omega_j.
\]

Let
\[
v_{kj} = \sum_i (u_{ki} + u \delta_{ki}) c(H_{ij} + H \delta_{ij}).
\]

By diagonalizing \((H_{ij} + H \delta_{ij})\), we may assume that \( v_{kj} \) and \( v_{jk} \) either have the same sign or are both equal zero. Then by putting (4.18) into (4.17),
\[
(4.19) \quad \int_{S^n} \langle X, e_{n+1} \rangle \left( \sum_{i \neq j} (v_{ii} v_{jj} - v_{ij} v_{ji}) \right) \det (H_{ij} + H \delta_{ij})^{-1} = 0.
\]

\( L_{HH}(u) = 0 \) implies that \( \Sigma_i v_{ii} = 0 \). Therefore,
\[
(4.20) \quad \sum_{i \neq j} (v_{ii} v_{jj} - v_{ij} v_{ji}) = \frac{1}{2} \left[ \left( \sum_i v_{ii} \right)^2 - \sum_i v_{ii}^2 \right] - \sum_{i \neq j} v_{ij} v_{ji} \leq 0.
\]

On the other hand, \( \det (H_{ij} + H \delta_{ij}) \) and \( \langle X, e_{n+1} \rangle = H \) are positive. Therefore,
(4.19) and (4.20) together show that \( u_{ij} = 0 \) for all \( i, j \). Hence

\[
\begin{aligned}
(4.21) & \quad u_{ij} + u \delta_{ij} = 0
\end{aligned}
\]

for all \( i, j \).

Putting (4.21) into (4.18), we find that \( Z = \text{const.} \) and \( u = \sum_{i=1}^{n+1} a_i x_i \) for some constants \( a_1, \ldots, a_{n+1} \).

Finally we prove the surjectivity of the linearized operator \( L_H \).

Let \( H^m(S^n) \) be the Sobolev space of functions defined on \( S^n \) with derivatives up to order \( m \). Then \( L_H \) can be considered as a bounded linear map from \( H^{k+2}(S^n) \) to \( H^k(S^n) \).

First of all, choose \( k \) so large that \( C^2(S^n) \to H^{k+2}(S^n) \). This is possible by the Sobolev lemma. With this choice, we can apply Lemma 5 and conclude that the kernel of \( L_H \) is given by the space of linear functions.

Therefore it follows from (4.2) that the kernel of the adjoint of \( L_H \) is the space of linear functions. Standard Hilbert space theory shows that, for every function \( f \) in \( H^k(S^n) \), we can find a solution \( u \in H^{k+2}(S^n) \) solving the equation \( L_H(u) = f \). The standard argument using the interior Schauder estimate also shows that, when \( f \in C^k(S^n) \), \( u \in C^{k+2}(S^n) \).

This finishes the proof of the surjectivity of \( L_H \) and hence the closedness of \( S_\alpha \) under the assumption that \( k \) is large enough to guarantee that \( C^2(S^n) \subset H^{k+2}(S^n) \). Therefore, for this \( k \), we have solved the equation \( \det(H_{ij} + H \delta_{ij}) = K^{-1} \) in the manner that, when \( K^{-1} \in C^{k,\alpha}(S^n) \) and verifies (2.1), \( H \in C^{k+2,\alpha}(S^n) \).

If we merely assume that \( K \) is a positive function in \( C^3(S^n) \), then we can approximate \( K \) in \( C^{2,1}(S^n) \)-norm by positive functions \( K_i \) in \( C^{k,\alpha}(S^n) \). In order to ensure the compatibility condition, we simply replace \( K_i \) by \( \tilde{K}_i \) so that

\[
\frac{1}{\tilde{K}_i} = 1 + \sum_{j=1}^{n} \left( \int_{S^n} x_j K_i^{-1} \right) \left( \int_{S^n} x_j^2 \right)^{-1} x_j + 2 \varepsilon_i,
\]

where

\[
-\varepsilon_i = \min \left( 0, \inf_{S^n} \left( \frac{1}{K_i} - \sum_{j=1}^{n} \left( \int_{S^n} x_j K_i^{-1} \right) \left( \int_{S^n} x_j^2 \right)^{-1} x_j \right) \right).
\]

Furthermore \( \int_{S^n} x_i \tilde{K}_i^{-1} = 0 \) and \( \{ \tilde{K}_i \} \to K \) in the \( C^{2,1}(S^n) \)-norm. By the above arguments, we can solve the equation \( \det(H_{ij} + H \delta_{ij}) = \tilde{K}_i^{-1} \) with solutions in \( C^{k+2,\alpha}(S^n) \). The arguments in proving the closedness of \( S_\alpha \) show that we can pick a convergent subsequence of these solutions to converge to a solution of
\[ \det (H_{ij} + H \delta_{ij}) = K^{-1}. \]

In conclusion, we have the following

**Theorem 1.** Let \( K \) be a positive function in \( C^k(S^n) \) with \( k \geq 3 \). Suppose \( \int_{S^n} x_i K^{-1} = 0 \) for all coordinate functions \( x_i \). Then we can solve the equation \( \det (H_{ij} + H \delta_{ij}) = K^{-1} \), where the solution belongs to \( C^{k+1, \alpha}(S^n) \) for all \( 1 > \alpha > 0 \). If \( K \) is analytic, then the solution is also analytic. Therefore we can find a compact strictly convex hypersurface in \( R^{n+1} \) whose support function is given by \( H \) and whose Gauss–Kronecker curvature function is \( K \). Moreover, any two such hypersurfaces must coincide after a translation.

**Proof:** Only the uniqueness part needs a proof. This is a consequence of Theorem 3 in Section 5.

### 5. Generalized Minkowski Problem

In this section, we shall briefly discuss the classical approach to the Minkowski problem.

Let \( M \) be a compact convex hypersurface in \( R^{n+1} \). Then the generalized Gauss map \( G : M \to S^n \) is a set-valued map which maps every \( x \in M \) to the set of all outward normals of the supporting planes of \( M \) passing through \( x \). From this map we can define a measure \( \mu(M) \) on \( S^n \) called the area function of \( M \) by setting

\[ \mu(M, E) = n\text{-dimensional volume of } \{ x \in M : G(x) \cap E \neq \emptyset \} \]

for any Borel subset \( E \) of \( S^n \).

(The complete additivity of \( \mu(M) \) follows from the fact that, at almost every point of \( M \), there is a unique supporting plane.)

When \( M \) is strictly convex and smooth, \( \mu(M) \) is the absolutely continuous measure on \( S^n \) represented by \( 1/K \), where \( K \) is the Gauss–Kronecker curvature of \( M \) transplanted to \( S^n \) via the Gauss map. When \( M \) is a polyhedron, \( \mu(M) \) is equal to \( \sum_{i=1}^l c_i \delta_{u_i} \), where \( \delta_{u_i} \) is the unit point mass at \( u_i \) and \( c_i \) is the \( n \)-dimensional measure of the face of \( M \) with outward normal equal to \( u_i \).

To study the convergence of compact convex sets, we define a metric \( \Delta \) on \( \mathcal{C} = \{ K : K \text{ is a compact convex set in } R^{n+1} \} \) in the following way:

\[ \Delta(K, K') = \sup_{x \in K'} d(x, K) + \sup_{x \in K} d(x, K') \]

for any \( K, K' \in \mathcal{C} \).
That \( \Delta \) is indeed a metric on \( \mathcal{C} \) can be easily verified.

**Proposition 1.** Suppose \( M, M_i \in \mathcal{C}, 1 \leq i < \infty \), and \( M_i \to M \) in the topology defined by \( \Delta \). Then \( \mu(M_i) \to \mu(M) \) weakly, i.e., for any \( f \in C(S^n) \),
\[
\int f \mu(M_i) \to \int f \mu(M).
\]

Proof: The proposition is a direct consequence of the following two lemmas.

**Lemma 8.** Suppose \( M, M_i \in \mathcal{C}, 1 \leq i < \infty \), and \( M_i \to M \). Suppose \( E \) is a closed subset of \( S^n \). Then
\[
\mu(M, E) \geq \limsup_{i \to \infty} \mu(M_i, E).
\]

**Lemma 9.** Suppose \( M, M_i \in \mathcal{C}, 1 \leq i < \infty \), and \( M_i \to M \). Suppose \( U \) is an open subset of \( S^n \). Then
\[
\mu(M, U) \leq \liminf_{i \to \infty} \mu(M_i, U).
\]

Proof of Lemma 8: Let \( G_i \) and \( G \) denote the Gauss map of \( M_i \) and \( M \), respectively. It is easy to see that \( G_i^{-1}(E) \) and \( G^{-1}(E) \) are closed sets. Set
\[
\lim_{i \to \infty} G_i^{-1}(E) = \{ x \in M : x \text{ is a limit point of some sequence } \{ x_i \} \text{ with } x_i \in G_i^{-1}(E) \}.
\]

To prove the assertion of Lemma 8, it suffices to show that for any open set \( U \) containing \( G^{-1}(E) \), \( \lim_{i \to \infty} G_i^{-1}(E) \) is a subset of \( U \). Suppose the contrary, let \( x \in \lim_{i \to \infty} G_i^{-1}(E) \setminus U \). Since \( x \notin U \), clearly \( G(x) \cap E = \emptyset \). However, \( x \in \lim_{i \to \infty} G_i^{-1}(E) \) implies that \( x \) is a limit point of some sequence \( \{ x_i \} \) with \( x_i \in G_i^{-1}(E) \). Since \( E \) is a closed subset of \( S^n \), we can conclude that \( x \in G^{-1}(E) \). This contradicts the fact that \( G(x) \cap E = \emptyset \), so \( U \) contains \( \lim_{i \to \infty} G_i^{-1}(E) \) and the proof of Lemma 8 is then completed.

Proof of Lemma 9: Notice that \( G_i^{-1}(U) \cap G_i^{-1}(S^n \setminus U) \) is a set of points with more than one supporting plane and hence are of zero \( n \)-dimensional measure. Thus,
\[
\liminf_{i \to \infty} \mu(M_i, U) = \liminf_{i \to \infty} (\mu(M_i, S^n) - \mu(M_i, S^n \setminus U))
\]
\[
= \liminf_{i \to \infty} \mu(M_i, S^n) - \limsup_{i \to \infty} \mu(M_i, S^n \setminus U)
\]
\[
\geq \mu(M, S^n) - \mu(M, S^n \setminus U)
\]
\[
\geq \mu(M, U).
\]
We introduce now briefly the concept of mixed volumes which is a natural and useful tool in studying the Minkowski problem.

For $K, K' \in \mathcal{C}$ and $t, t' \geq 0$, we define

$$g(t, t', K, K') = V_{n+1}(tK + t'K'),$$

where $V_{n+1}(tK + t'K')$ denotes the $(n+1)$-dimensional measure of the set \( \{tx + t'x' : x \in K, x' \in K' \} \). Then it is easy to see that $g$ is a continuous function defined on \([0, \infty) \times [0, \infty) \times \mathcal{C} \times \mathcal{C}\).

**Proposition 2.** Let $K, K' \in \mathcal{C}$. Then there are non-negative constants denoted by $V(K, k; K', n+1-k)$, $0 \leq k \leq n+1$, such that

$$g(t, t', K, K') = \sum_{k=0}^{n+1} \binom{n+1}{k} t^k t'^{n+1-k} V(K, k; K', n+1-k).$$

**Proof:** We proceed by induction on the dimension. When the dimension is one, the assertion is obvious.

Suppose the proposition is true for dimension $n$. We are going to show that the proposition is also true for dimension $n+1$. Suppose $K, K'$ are polyhedrons. Let $\bar{K} = tK + t'K$. Then $\bar{K}$ is also a polyhedron. Since $g(t, t', K, K') = g(t, t', K + x, K' + x')$ for any $x \in K$, $x' \in K'$, we may suppose $K$, $K'$ contains the origin. Let $P_1, \cdots, P_l$ be $n$-dimensional hyperplanes determined by the $n$-dimensional faces of $\bar{K}$, and let $u_i$ be the outward normal of $P_i$. Let $P_i$, $P_i'$, $1 \leq i \leq l$, be supporting planes of $K$, $K'$ with outward normal $u_i$. Then it is easy to verify that

$$P_i \cap \bar{K} = t(P_i \cap K) + t'(P_i' \cap K'), \quad 1 \leq i \leq l.$$  \hspace{1cm} (5.1)

Let $H, H', \bar{H}$ denote the support functions of $K$, $K'$, $\bar{K}$, respectively. By our assumption, $H, H', \bar{H}$ are all non-negative. Equation (3.4) shows that

$$V_{n+1}(\bar{K}) = \frac{1}{n+1} \sum_{i=1}^{l} \bar{H}(u_i) V_n(P_i \cap \bar{K}), \hspace{1cm} \text{where} \hspace{1cm} V_n(P \cap K) \text{ denotes the } n\text{-dimensional measure of } P_i \cap K. \text{ It is easy to see that } \bar{H}(u_i) = tH(u_i) + t'H'(u_i). \text{ Applying (5.1) to (5.2) we obtain}$

$$V_{n+1}(\bar{K}) = \frac{1}{n+1} \sum_{i=1}^{l} (tH(u_i) + t'H'(u_i)) V_n(t(P_i \cap K) + t'(P_i' \cap K')). \hspace{1cm} (5.3)$$
Now we see that the assertion of the proposition is valid when we apply the inductive assumption to $V_n(t(P_i \cap K) + t'(P'_i \cap K'))$. Let $K, K' \in \mathcal{C}$ which are not necessarily polyhedrons. Let $\{K_i\}, \{K'_i\}$ be polyhedrons such that $K_i \to K$ and $K'_i \to K$. Then

$$g(t, t', K_i, K'_i) \to g(t, t', K, K')$$ as $i \to \infty$.

We can enclose all $K_i, K'_i$ in a large ball $B$. Therefore,

$$g(t, t', K_i, K'_i) \leq g(t, t', B, B) = (t + t')^{n+1} V_{n+1}(B).$$

Thus $\left(\binom{n+1}{k}\right) V(K_i, k; K'_i, n+1-k) \leq 2^{n+1} V_{n+1}(B)$, for $1 \leq i < \infty$ and $0 \leq k \leq n+1$.

We may suppose without loss of generality that $V(K_i, k; K'_i, n+1-k)$ converges as $i \to \infty$. We then conclude that $g(t, t', K, K')$ can be represented as a homogeneous polynomial of degree $n+1$ with non-negative coefficients. This finishes the proof of Proposition 2.

**DEFINITION.** $V(K, k; K', n+1-k)$ is called the mixed volume of $K, K'$, $0 \leq k \leq n+1$.

We then derive an important integral formula for $V(K, n; K', 1)$.

**PROPOSITION 3.** Let $K, K' \in \mathcal{C}$ and let $H'$ be the support function of $K'$. Then

$$V(K, n; K', 1) = \frac{1}{n+1} \int H' \, d\mu(K).$$

**Proof:** Suppose $K, K'$ are polyhedrons and they both contain the origin. Let $D(K')$ denote the extrinsic diameter of $K'$. Observe that

$$V(K, n; K', 1) = \frac{1}{n+1} \lim_{t \to 0} \frac{V_{n+1}(K + tK') - V_{n+1}(K)}{t}.$$

Let $P_1, \cdots, P_l$ be $n$-dimensional hyperplanes determined by $n$-dimensional faces of $K$ and let $u_1, \cdots, u_l$ be their outward normals. It is obvious that

$$K + tK' \supseteq K \cup \bigcup_{i=1}^{l} \{x + tu_i : x \in P_i \cap K, 0 \leq t \leq H(u_i)\}.$$
Hence,

\[ V_{n+1}(K + tK') \equiv V_{n+1}(K) + \sum_{i=1}^{l} tH(u_i) V_n(P_i \cap K). \]

On the other hand, if we define \( A \) to be the set of all points whose distance from the \((n-1)\)-skeleton of \( K \) is less than \( tD(K') \), then

\[ K + tK' \subset K \cup A \cup \bigcup_{i=1}^{l} \{ x + tu_i : x \in P_i \cap K, 0 \leq t \leq H(u_i) \}. \]

Consequently,

\[ V_{n+1}(K + tK') \leq V_{n+1}(K) + \sum_{i=1}^{l} tH(u_i) V_n(P_i \cap K) + O(t^2). \]

Thus (5.4) is valid when \( K, K' \) are polyhedrons. The general case is then a consequence of polyhedron approximations and Proposition 1.

**Corollary 1.** Suppose \( K_i \in \mathcal{C} \) and \( \mu(K_i) \) converges weakly to another Borel measure \( \mu \) on \( S^n \). Then, after translations of the \( K_i \), a subsequence of them converges to a compact convex set \( K \in \mathcal{C} \) with \( \mu(K) = \mu \).

**Proof:** For any \( u \in S^n \), let \( L_u = \{ tu : 0 \leq t \leq 1 \} \). For any compact convex set \( K' \in \mathcal{C} \), \( V(K', n; L_u, 1) \) is the \( n \)-dimensional measure of the projection of \( K' \) into a hyperplane with normal \( u \). Equation (5.4) then implies that we can estimate an upper bound of \( V(K_i, n; L_u, 1) \) for all \( u \in S^n \). As in Lemma 3, we can then estimate a uniform bound for the extrinsic diameter of each \( K_i \). Translate each \( K_i \) so that its center of gravity locates at the origin. Since we have a uniform bound for the extrinsic diameter of each \( K_i \), the Blaschke selection theorem guarantees that a subsequence of \( \{ K_i \} \) will converge to some \( K \in \mathcal{C} \). That \( \mu(K) = \mu \) is a direct consequence of Proposition 1.

We now state the well known Brunn–Minkowski inequality (for a simple proof, see [1], [5]).

**Theorem 2.** Suppose \( K, K' \in \mathcal{C} \). Then the function \( h(t) = g(t, 1-t, K, K')^{1/(n+1)} \), \( t \in [0, 1] \), is a concave function in \( t \). If \( K, K' \) do not lie on parallel hyperplanes, \( h(t) \) is linear if and only if \( K \) and \( K' \) are homothetic.

**Corollary 2.** Suppose \( K, K' \in \mathcal{C} \). Then

\[ (V(K, n; K', 1))^{n+1} \geq (V_{n+1}(K))^{n} V_{n+1}(K'). \]
If $K$, $K'$ do not lie on parallel hyperplanes, equality in (5.5) holds if and only if $K$ and $K'$ are homothetic.

We can then prove the uniqueness of the Minkowski problem.

**Theorem 3.** Suppose that $K, K'$ are elements of $\mathcal{C}$ and $\mu(K) = \mu(K')$. If $\mu(K, \{x \in S^n : \langle x, u \rangle > 0\}) > 0$ for all $u \in S^n$, then $K$ and $K'$ coincide up to a translation.

**Proof:** Let $u \in S^n$ and $L_u = \{tu : 0 \leq t \leq 1\}$. Then the support function of $L_u$, denoted by $T_u$, is

$$T_u(w) = \max (0, u \cdot w), \quad w \in S^n.$$ 

Thus

$$V(K, n; L_u, 1) = \frac{1}{n+1} \int T_u(w) \, d\mu(K) > 0 \text{ for all } u \in S^n.$$ 

However, $V(K, n; L_u, 1)$ is the $n$-dimensional measure of the projection of $K$ into a hyperplane with normal $u$. This shows that $K$ does not lie on any hyperplane. Let $H, H'$ denote the support function of $K, K'$, respectively.

Then by (5.4),

$$V(K, n; K', 1) = \frac{1}{n+1} \int H' \, d\mu(K) = \frac{1}{n+1} \int H' \, d\mu(K') = V_{n+1}(K').$$

So an application of (5.5) shows that

$$V_{n+1}(K') \geq V_{n+1}(K).$$

Similarly, $V_{n+1}(K) \geq V_{n+1}(K')$ and hence $V_{n+1}(K) = V_{n+1}(K')$. Therefore, $(V(K, n; K', 1))^{n+1} = (V_{n+1}(K))^{n} V_{n+1}(K')$ and then $K$ and $K'$ must be homothetic. However, $K$ and $K'$ enclose the same volume. This shows that $K$ and $K'$ coincide up to a translation.

Let $K \in \mathcal{C}$. Then $\mu(K)$ will satisfy certain “compatibility” conditions.

**Lemma 10.** Let $K \in \mathcal{C}$. Then $\int x_i \, d\mu(K) = 0$ for each coordinate function $x_i$ on $S^n$.

**Proof:** It suffices to prove the lemma for polyhedrons. The general case follows by polyhedron approximations and Proposition 1.

Suppose $K$ is a polyhedron. Let $a = (a_1, \ldots, a_{n+1})$ be any constant vector in $R^{n+1}$. Then, since $\Delta(\sum_{i=1}^{n+1} a_i x_i) = 0$ in $R^{n+1}$, we obtain, by the divergence
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formula,

\[ \int_{\partial K} \langle a, n \rangle = 0, \]

where \( n \) is the outward unit normal of \( K \). Transforming (5.6) to an equation on \( S^n \), we prove the assertion of the lemma.

The existence theorem of Minkowski, Alexandrov, Fenchel and Jessen shows that the converse of Lemma 10 is almost true (see [2]).

**Theorem 4.** Let \( \mu \) be a non-negative Borel measure on \( S^n \) such that

\[ \int_{S^n} x_i \mu = 0 \text{ for all } i \text{ and } \mu(\{x : \langle x, u \rangle > 0\}) > 0 \text{ for all } u \in S^n. \]

Then there exists \( K \in \mathcal{C} \) such that \( \mu = \mu(K) \). Furthermore, \( K \) is unique up to translations.

**Remark.** As in the proof of Theorem 3, one imposes the condition \( \mu(\{x : \langle x, u \rangle > 0\}) > 0 \) for \( u \in S^n \) is to guarantee that the sought after convex body has \( " \text{dimension} " \ n + 1 \).

We only sketch the proof of Theorem 4.

In our case, we can give a complete proof of Theorem 4 when \( \mu \) is absolutely continuous.

Let \( \mu \) be represented by a non-negative measurable function \( f \) on \( S^n \). It is easy to find positive smooth functions \( K_i \) on \( S^n \) such that \( 1/K_i \to f \) in the \( L^1 \) sense and \( \int_{S^n} x_i (1/K_i) = 0 \) for \( 1 \leq j \leq n + 1, 1 \leq i < \infty \). Using Theorem 1 we can find smooth convex bodies \( M_i \in \mathcal{C} \) such that \( \mu(M_i) \) is represented by \( 1/K_i \). Corollary 1 shows that there exists \( M \in \mathcal{C} \) with \( \mu(M) \) an absolutely continuous measure represented by \( f \). This solves the Minkowski problem when \( \mu \) is absolutely continuous. The proof of Theorem 4 is actually analogous to the one given above. We first solve the case when \( \mu \) is a sum of point masses. (The solution will be a polyhedron.) For the general case, we approximate \( \mu \) weakly by point masses \( \mu_i \). Applying Corollary 1 to the solutions of \( \mu_i \), we obtain \( M \in \mathcal{C} \) with \( \mu(M) = \mu \).

**Bibliography**


[7] Pogorelov, A. V., *On the regularity of generalized solutions of the equation* \( \det (\partial^2 u/\partial x_i \partial x_j) = \Phi(x_1, \cdots, x_n) > 0 \), Soviet Math. Dokl. Vol. 12, 1971, pp. 1436–1440.


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