SECOND ORDER ESTIMATES AND REGULARITY FOR
FULLY NONLINEAR ELLIPTIC EQUATIONS
ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We derive a priori second order estimates for solutions of a class of fully nonlinear elliptic equations on Riemannian manifolds under structure conditions which are close to optimal. We treat both equations on closed manifolds, and the Dirichlet problem on manifolds with boundary without any geometric restrictions to the boundary. These estimates yield regularity and existence results some of which are new even for equations in Euclidean space.

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1. Introduction

Let \((M^n, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\) with smooth boundary \(\partial M, \bar{M} := M \cup \partial M\), and \(\chi\) a smooth \((0, 2)\) tensor on \(\bar{M}\). In this paper we consider fully nonlinear equations of the form

\[
(1.1) \quad f(\lambda[\nabla^2 u + \chi]) = \psi \text{ in } M
\]

where \(\nabla^2 u\) denotes the Hessian of \(u \in C^2(M)\) and \(\lambda[\nabla^2 u + \chi] = (\lambda_1, \cdots, \lambda_n)\) are the eigenvalues of \(\nabla^2 u + \chi\) with respect to the metric \(g\), and \(f\) is a smooth symmetric function defined in a symmetric open and convex cone \(\Gamma \subseteq \mathbb{R}^n\) with vertex at the origin and boundary \(\partial \Gamma \neq \emptyset\),

\[
(1.2) \quad \Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} \subseteq \Gamma.
\]

The study of fully nonlinear equations of form (1.1) were pioneered by Ivochkina [25] who treated some special cases, and Caffarelli-Nirenberg-Spruck [5] in \(\mathbb{R}^n\) (with \(\chi = 0\))

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and have received extensive attention since then. The standard and fundamental structure conditions on \( f \) in the literature include

\[
(1.3) \quad f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in} \quad \Gamma, \quad 1 \leq i \leq n,
\]

\[
(1.4) \quad f \text{ is a concave function in } \Gamma,
\]

and

\[
(1.5) \quad \delta_{\psi,f} \equiv \inf_M \psi - \sup_{\partial \Gamma} f > 0
\]

where

\[
\sup_{\partial \Gamma} f \equiv \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda).
\]

Condition (1.3) ensures that equation (1.1) is elliptic for solutions \( u \in C^2(M) \) with \( \lambda[\nabla^2 u + \chi] \in \Gamma \); we shall call such functions \textit{admissible}, while condition (1.4) implies the function \( F \) defined by \( F(A) = f(\lambda[A]) \) to be concave for \( A \in S^{n\times n} \) with \( \lambda[A] \in \Gamma \), where \( S^{n\times n} \) is the set of \( n \times n \) symmetric matrices; see [5].

The most typical equations of form (1.1) are given by \( f = \sigma_1^k \) and \( f = (\sigma_k / \sigma_l)^{k-l} \), \( 1 \leq l < k \leq n \) defined on the cone

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for } 1 \leq j \leq k \},
\]

where \( \sigma_k \) is the \( k \)-th elementary symmetric function

\[
\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n.
\]

These functions satisfy (1.3)-(1.4) and have other important properties which have been widely used in study of the corresponding equations; see e.g. [5], [7], [25], [26], [27], [29], [31], [33], [36], [40].

Geometric quantities of the form \( \nabla^2 u + \chi \) appear naturally in many different subjects. For example, \( \text{Ric}_u = \nabla^2 u + \text{Ric} \), where \( \text{Ric} \) is the Ricci tensor of \( (M^n, g) \), is the Bakry-Emery Ricci tensor of the Riemannian measure space \( (M^n, g, e^{-u}d\text{Vol}_g) \) (see e.g. [41] and references therein), while the gradient Ricci soliton equation takes the form

\[
\nabla^2 u + \text{Ric} = \lambda g
\]

which plays important roles in the theory of Ricci flow and has received extensive study. When \( \chi = g \) equation (1.1) was studied by Li [31] and Urbas [39] on closed manifolds.
It is well understood that in order to solve equation (1.1) a central issue is to derive a priori $C^2$ estimates for admissible solutions; the Evans-Krylov theorem then yields $C^{2,\alpha}$ bounds by assumptions (1.4) and (1.5) which implies that equation (1.1) becomes uniformly elliptic once a priori $C^2$ bounds are established for admissible solutions.

Our main concern in this paper is the estimates for second derivatives of admissible solutions

$$
|\nabla^2 u| \leq C \text{ in } \bar{M}
$$

where $C$ may depend on $|u|_{C^1(\bar{M})}$.

Such estimates have been established under various assumptions on $f$ in addition to (1.3)-(1.5) as well as conditions on the geometry of $\partial M$, with significant contributions from Caffarelli-Nirenberg-Spruck [5], Ivochkina [26], Li [31], Trudinger [37], Urbas [39], and Ivochkina-Trudinger-Wang [28] who considered the degenerate case, among others (see also [11], [14]). Our primary goal in this paper is to establish (1.6) under conditions (in addition to (1.3)-(1.5)) which are essentially optimal, on general Riemannian manifolds with (smooth, compact but otherwise) arbitrary boundary. (In a forthcoming paper we shall come back to discuss possibilities to weaken (1.4), the concavity condition.) In order to state our result we first introduce some notation.

For $\sigma > \sup_{\partial M} f$, define $\Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \}$, and we shall only consider the case $\Gamma^\sigma \neq \emptyset$. By conditions (1.3) and (1.4) we see that the boundary of $\Gamma^\sigma$

$$
\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \}
$$

is a smooth convex hypersurface in $\mathbb{R}^n$. Define for $\mu \in \Gamma \setminus \Gamma^\sigma$

$$
S^\sigma_\mu = \{ \lambda \in \partial \Gamma^\sigma : \mu \in T_\lambda \partial \Gamma^\sigma \}
$$

where $T_\lambda \partial \Gamma^\sigma$ denotes the tangent plane of $\partial \Gamma^\sigma$ at $\lambda$, and

$$
\mathcal{V}_\sigma = \{ \mu \in \Gamma \setminus \Gamma^\sigma : S^\sigma_\mu \neq \emptyset \},
$$

$$
\mathcal{V}_\sigma^+ = \{ \mu \in \mathcal{V}_\sigma : S^\sigma_\mu \text{ is compact} \},
$$

$$
\mathcal{C}^+_\sigma = \mathcal{V}_\sigma^+ \cup \Gamma^\sigma.
$$

Note that $\mathcal{C}^+_\sigma$ is an open subset of $\Gamma$. For convenience we call $\mathcal{C}_\sigma := \partial \mathcal{C}^+_\sigma$ the tangent cone at infinity of $\Gamma^\sigma$.

Our first result may be stated as follows.

**Theorem 1.1.** Let $\psi \in C^2(M) \cap C^1(\bar{M})$ and $u \in C^4(M) \cap C^2(\bar{M})$ be an admissible solution of (1.1). Assume (1.3)-(1.5) hold and that there exists a function $u \in C^2(\bar{M})$...
satisfying for all \( x \in \bar{M} \)

\[ (1.7) \quad \lambda[\nabla^2 u + \chi](x) \in C^+_{\psi(x)}. \]

Then

\[ (1.8) \quad \max_M |\nabla^2 u| \leq C_1 \left( 1 + \max_{\partial M} |\nabla^2 u| \right). \]

where \( C_1 \) depends on \( |u|_{C^1(\bar{M})} \) (but not on \( \delta_{\psi,f} \)). In particular, if \( M \) is closed, i.e. \( \partial M = \emptyset \) then

\[ (1.9) \quad |\nabla^2 u| \leq C_1 e^{C_2(u-\inf_M u)} \quad \text{on } M \]

where \( C_2 \) is a uniform constant (independent of \( u \)).

Equation (1.1) with \( \chi = g \) was first studied by Li [31] who derived (1.6) on closed manifolds of nonnegative sectional curvature, followed by Urbas [39] who removed the nonnegative curvature assumption. In addition to (1.3)-(1.5), both of these papers needed extra assumptions which exclude the case \( f = (\sigma_k/\sigma_l)^{1/(k-l)} \). This case is covered by Theorem 1.1; see [15] where we also show that the hypotheses in [31] implies that (1.7) for \( \chi = g \) is satisfied by any constant.

From Theorem 1.1 and the Evans-Krylov theorem we obtain the following regularity result by approximation.

**Theorem 1.2.** Let \((M^n, g)\) be a closed Riemannian manifold and \( \psi \in C^{1,1}(M) \). Under conditions (1.3)-(1.5) and (1.7), any weak admissible solution (in the viscosity sense) \( u \in C^{0,1}(M) \) of (1.1) belongs to \( C^{2,\alpha}(M) \), \( 0 < \alpha < 1 \) and (1.9) holds.

Higher regularities follow from the classical Schauder theory for linear elliptic equations. In particular, \( u \in C^\infty(M) \) if \( \psi \in C^\infty(M) \).

**Remark 1.3.** If \( \chi \in C^+_\sigma \) for all \( \sigma \) (e.g. if \( \chi = ag, a > 0 \) and \( 0 \in \overline{C^+_\sigma} \)), any constant \( u \) satisfies (1.7). For \( f = \sigma_k^{1/k} (k \geq 2), \Gamma_n \subset C^+_\sigma \) for any \( \sigma > 0 \).

**Corollary 1.4.** Let \((M, g)\) be a closed Riemannian manifold and \( \psi \in C^{1,1}(M) \). In addition to (1.3)-(1.5), suppose \( \chi \in C^+_\sigma \) for all \( \sup_{\partial M} f < \sigma \leq \sup_M \psi \). Then any admissible weak solution \( u \in C^{0,1}(M) \) of (1.1) belongs to \( C^{2,\alpha}(M) \), \( 0 < \alpha < 1 \), and (1.9) holds.

We now turn to the second order boundary estimate

\[ (1.10) \quad \max_{\partial M} |\nabla^2 u| \leq C_3 \]
when $\partial M \neq \emptyset$. Even for domains in $\mathbb{R}^n$ this is very subtle and usually requires extra assumptions on $f$ and $\partial M$. In their original work [5], Caffarelli-Nirenberg-Spruck derived (1.10) for the Dirichlet problem (with $\chi = 0$) in a bounded domain $M \subset \mathbb{R}^n$ satisfying the curvature condition: there exists $R > 0$ such that

$$\sum f_i \lambda_i \geq 0 \quad \text{in } \Gamma \cap \{ \inf_M \psi \leq f \leq \sup_M \psi \}. \quad (1.14)$$

Their result was extended by Li [31] to the general case where $\chi$ is a symmetric matrix, and by Trudinger [37] who removed condition (1.12).

It was shown in [5] that if a domain $\Omega$ of type 2 in $\mathbb{R}^n$ satisfies (1.11) then $\partial \Omega$ is connected; see [5] for details. In particular, for $\Gamma = \Gamma_n$, (1.11) implies that $M \subset \mathbb{R}^n$ is strictly convex. In this paper we wish to derive the boundary estimate (1.10) on a general Riemannian manifold $M$ without imposing any geometric restrictions to $\partial M$ beyond being smooth and compact. The first effort in this direction was made by the author [11] where we had to assume (1.12) and the existence of an admissible strict subsolution; Trudinger [37] later showed that one can replace (1.12) by (1.13). In this paper we were able to prove the following result.

**Theorem 1.5.** Let $\psi \in C^1(\bar{M})$, $\varphi \in C^4(\partial M)$ and $u \in C^3(M) \cap C^2(\bar{M})$ be an admissible solution of (1.1) with $u = \varphi$ on $\partial M$. Assume $f$ satisfies (1.3)-(1.5) and

$$\sum f_i \lambda_i \geq 0 \quad \text{in } \Gamma \cap \{ \inf_M \psi \leq f \leq \sup_M \psi \}. \quad (1.14)$$

Suppose that there exists an admissible subsolution $u \in C^0(\bar{M})$ in the viscosity sense:

$$\begin{cases} f(\lambda [\nabla^2 u + \chi]) \geq \psi & \text{in } \bar{M}, \\ u = \varphi & \text{on } \partial M, \end{cases} \quad (1.15)$$

and that $u$ is $C^2$ and satisfies (1.7) in a neighborhood of $\partial M$. Then the boundary estimate (1.10) holds with $C_3$ depending on $\|u\|_{C^1(\bar{M})}$ and $\delta^{-1}_{\psi,f}$. 
Remark 1.6. An admissible subsolution $u \in C^2(\bar{M})$ will automatically satisfy (1.7) provided that $V_\alpha^+ = V_\sigma$ which is equivalent to
\begin{equation}
\partial \Gamma^\sigma \cap \partial C^+_\sigma = \emptyset, \forall \sigma \in \left[ \inf_M \psi, \sup_M \psi \right].
\end{equation}
Condition (1.16) excludes the linear function $f = \sigma_1$ which corresponds to the Poisson equation, but is satisfied by a wide class of concave functions including $f = \sigma_1^{1/k}$, $k \geq 2$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ for all $1 \leq l < k \leq n$; see Lemma 2.10. Note that condition (1.16) holds if $\partial \Gamma^\sigma$ is strictly convex at infinity, i.e. outside a compact set.

Remark 1.7. For bounded domains in $\mathbb{R}^n$ and any given smooth boundary data it was shown in [5] that (1.11) and (1.13) imply the existence of admissible strict subsolutions which satisfy condition (1.7) automatically.

Remark 1.8. The hypothesis (1.14), which is clearly implied by (1.13), can be dropped when $M$ is a bounded domain in $\mathbb{R}^n$. Since it requires different arguments in several places of the proof, we shall treat it elsewhere ([15]).

Remark 1.9. If $\partial M \neq \emptyset$ and there is a strictly convex function $v \in C^2(\bar{M})$, then $u = Av$ satisfies (1.7) on $\bar{M}$ for $A$ sufficiently large.

The Dirichlet problem for equation (1.1) in $\mathbb{R}^n$ was treated by Caffarelli-Nirenberg-Spruck [5], followed by [31], [11], [37], [28] among others. The important special case $f = \sigma_1^k$ has also received extensive study from different aspects, including [30], [26], [40], [38], [7]. Applying Theorems 1.1 and 1.5 one can prove the following existence result by the standard continuity method.

**Theorem 1.10.** Let $\psi \in C^\infty(\bar{M})$ and $\varphi \in C^\infty(\partial M)$. Suppose $f$ satisfies (1.3)-(1.5), (1.14) and that there exists an admissible subsolution $u \in C^2(\bar{M})$ satisfying (1.15) and (1.7) for all $x \in \bar{M}$. Then there exists an admissible solution $u \in C^\infty(\bar{M})$ of the Dirichlet problem for equation (1.1) with boundary condition $u = \varphi$ on $\partial M$, provided that (i) $\Gamma = \Gamma_n$, or (ii) the sectional curvature of $(M, g)$ is nonnegative, or (iii) $f$ satisfies
\begin{equation}
f_j \geq \delta_0 \sum f_i(\lambda) \text{ if } \lambda_j < 0, \text{ on } \partial \Gamma^\sigma \forall \sigma > \sup_{\partial \Gamma} f.
\end{equation}

For bounded domains in $\mathbb{R}^n$, Theorem 1.10 holds without assumption (1.14) and extends the previous results of Caffarelli-Nirenberg-Spruck [5], Trudinger [37] and the author [11]. The assumptions (i)-(iii) are only needed to derive gradient estimates; see Proposition 5.1.
Corollary 1.11. Let \( f = \sigma_k^{1/k} \), \( k \geq 2 \) or \( f = (\sigma_k/\sigma_l)^{1/(k-l)}, \) \( 1 \leq l < k \leq n \). Given \( \psi \in C^\infty(\bar{M}), \psi > 0 \) and \( \varphi \in C^\infty(\partial M) \), suppose that there exists an admissible subsolution \( \underline{u} \in C^2(\bar{M}) \) satisfying (1.15). Then there exists an admissible solution \( u \in C^\infty(\bar{M}) \) of equation (1.1) with \( u = \varphi \) on \( \partial M \).

For \( f = (\sigma_k/\sigma_l)^{1/(k-l)}, \) \( 0 \leq l < k \leq n \), which satisfies (1.17), Corollary 1.11 is new even when \( M \) is a bounded domain in \( \mathbb{R}^n \).

It would be worthwhile to note that in Theorem 1.10, since there are no geometric restrictions to \( \partial M \) being made, the Dirichlet problem is not always solvable without the subsolution assumption. In the classical theory of elliptic equations, a standard technique is to use the distance function to the boundary to construct local barriers for boundary estimates. So one usually need require the boundary to possess certain geometric properties; see e.g. [35] for the prescribed mean curvature equation and [4], [3] for Monge-Ampère equations; see also [10] and [5]. In our approach we use \( u - \underline{u} \) to replace the boundary distance function in deriving the second order boundary estimates. This idea was first used by Haffman-Rosenberg-Spruck [24] and further developed in [18], [16], [12], [13] to treat the real and complex Monge-Ampère equations in general domains as well as in [11], [14] for more general fully nonlinear equations. The technique has found some useful applications; see e.g. [2], [6], [18], [19], [20], [22], [34].

We shall also make use of \( u - \underline{u} \) in the proof of the global estimate (1.8). This is one of the feature marks of the paper; see the proof in Section 3. Note that in Theorem 1.1 the function \( \underline{u} \) is not necessarily a subsolution. On a closed manifold, an admissible subsolution of equation (1.1) must be a solution if there is a solution at all, and any two admissible solutions differ at most by a constant. This is a consequence of the concavity condition (1.4) and the maximum principle.

The rest of this paper is organized as follows. In Section 2 we discuss some consequences of the concavity condition. Our proof of the estimates heavily depends on results in Section 2 where we also give a brief proof that (1.16) holds for \( f = \sigma_k^{1/k}, \) \( k \geq 2 \) and \( f = (\sigma_k/\sigma_l)^{1/(k-l)}, \) \( 1 \leq l < k \leq n \), and therefore Corollary 1.11 follows from Theorem 1.10. In Sections 3 and 4 we derive the global and boundary estimates for second derivatives, respectively. In Section 5 we derive the gradient estimates needed to prove Theorem 1.10.
The author wishes to thank Jiaping Wang for helpful discussions on the proof of Theorem 2.5 and related topics, and the referees for their insightful comments and suggestions.

Finally, we remark that the ideas in this article can be used to significantly weaken the concavity assumption (1.4) and treat general fully nonlinear elliptic equations of the form

\[ F(\nabla^2 u + \chi) = \psi. \]

This will be addressed in forthcoming papers.

2. The concavity condition

In this section we examine the properties of \( \Gamma^\sigma \) and the associated cone \( C_\sigma^+ \) on which will be based our proof of the estimates (1.8) and (1.10). We shall have more detailed discussions in [15].

Let \( \sigma > \sup_{\partial \Gamma} f \) and assume \( \Gamma^\sigma := \{ f > \sigma \} \neq \emptyset \). Then \( \partial \Gamma^\sigma \) is a smooth convex noncompact complete hypersurface contained in \( \Gamma \). Clearly \( \Gamma^\sigma \neq C_\sigma^+ \) unless \( \partial \Gamma^\sigma \) is a plane.

Let \( \mu, \lambda \in \partial \Gamma^\sigma \). By the convexity of \( \partial \Gamma^\sigma \), the open segment

\[ (\mu, \lambda) := \{ t\mu + (1-t)\lambda : 0 < t < 1 \} \]

is completely contained in either \( \partial \Gamma^\sigma \) or \( \Gamma^\sigma \) by condition (1.3). Therefore,

\[ f(t\mu + (1-t)\lambda) > \sigma, \quad \forall 0 < t < 1 \]

unless \((\mu, \lambda) \subset \partial \Gamma^\sigma\).

For \( R > |\mu| \), let

\[ \Theta_R(\mu) := \inf_{\lambda \in \partial B_R(0) \cap \partial \Gamma^\sigma} \max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda) - \sigma \geq 0. \]

Clearly \( \Theta_R(\mu) = 0 \) if and only if \((\mu, \lambda) \subset \partial \Gamma^\sigma\) for some \( \lambda \in \partial B_R(0) \cap \partial \Gamma^\sigma \), since the set \( \partial B_R(0) \cap \partial \Gamma^\sigma \) is compact.

**Lemma 2.1.** For \( \mu \in \partial \Gamma^\sigma \), \( \Theta_R(\mu) \) is nondecreasing in \( R \). Moreover, if \( \Theta_{R_0}(\mu) > 0 \) for some \( R_0 \geq |\mu| \) then \( \Theta_{R'} > \Theta_R \) for all \( R' > R \geq R_0 \).
Proof. Write $\Theta_R = \Theta_R(\mu)$ when there is no possible confusion. Suppose $\Theta_{R_0}(\mu) > 0$ for some $R_0 \geq |\mu|$. Let $R' > R \geq R_0$ and assume $\lambda_{R'} \in \partial B_{R'}(0) \cap \partial \Gamma^\sigma$ such that

$$\Theta_{R'} = \max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda_{R'}) - \sigma.$$ 

Let $P$ be the (two dimensional) plane through $\mu, \lambda_{R'}$ and the origin of $\mathbb{R}^n$. There is a point $\lambda_R \in \partial B_R(0)$ which lies between $\mu$ and $\lambda_{R'}$ on the curve $P \cap \partial \Gamma^\sigma$. Note that $\mu, \lambda_R$ and $\lambda_{R'}$ are not on a straight line, for $(\mu, \lambda_R)$ cannot be part of $(\mu, \lambda_{R'})$ since $\Theta_{R_0} > 0$ and $\partial \Gamma^\sigma$ is convex. We see that

$$\max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda_R) - \sigma < \Theta_{R'}$$

by condition (1.3). This proves $\Theta_R < \Theta_{R'}$. \qed

**Corollary 2.2.** Let $\mu \in \partial \Gamma^\sigma$. The following are equivalent:

(a) $\mu \in \partial \mathcal{C}^+_\sigma$;

(b) $\Theta_R(\mu) = 0$ for all $R > |\mu|$;

(c) $\partial \Gamma^\sigma \cap \partial \mathcal{C}^+_\sigma$ contains a ray through $\mu$;

(d) $T_\mu \partial \Gamma^\sigma \cap \partial \mathcal{C}^+_\sigma$ contains a ray through $\mu$.

**Lemma 2.3.** Let $\mu \in \Gamma^\sigma \setminus \partial \mathcal{C}^+_\sigma$. There exist positive constants $\omega_\mu, N_\mu$ such that for any $\lambda \in \partial \Gamma^\sigma$, when $|\lambda| \geq N_\mu$,

$$\sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) \geq \omega_\mu. \quad (2.1)$$

Proof. By the concavity of $f$,

$$\sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda).$$

We see (2.1) holds if $f(\mu) > \sigma$. So we assume $\mu \in \partial \Gamma^\sigma$. By Corollary 2.2, $\Theta_R(\mu) > 0$ for all $R$ sufficiently large, and therefore, again by the concavity of $f$,

$$\sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) \geq \max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda) - \sigma \geq \Theta_R(\mu) > 0$$

for any $\lambda \in \partial B_R(0) \cap \partial \Gamma^\sigma$. Since $\Theta_R(\mu)$ is increasing in $R$, we see that Lemma 2.3 holds. \qed

**Lemma 2.4.** Let $K$ be a compact subset of $\Gamma^\sigma \setminus \partial \mathcal{C}^+_\sigma$. There exist positive constants $\omega_K, N_K$ such that for any $\lambda \in \partial \Gamma^\sigma$, when $|\lambda| \geq N_K$,

$$\sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) \geq \omega_K, \forall \mu \in K. \quad (2.2)$$
Proof. Let \( d := \text{dist}(K, \partial C^+_\sigma) \) denote the distance from \( K \) to \( \partial C^+_\sigma \). By the assumption we see that \( K \) is a compact subset of \( C^+_\sigma \) and therefore \( d > 0 \). This implies that there exists \( R_0 \) sufficiently large such that
\[
\text{dist}(K, T_\lambda \partial \Gamma^\sigma) \geq \frac{d}{2}, \quad \forall \lambda \in \partial \Gamma^\sigma \cap \partial B_{R_0}(0).
\]

Since \( f \) is continuous, we have
\[
\omega_K := \inf_{\mu \in K} \Theta_{R_0}(\mu) = \inf_{\mu \in K} \inf_{\lambda \in \partial B_{R_0}(0) \cap \partial \Gamma^\sigma} \max_{0 \leq t \leq 1} f(t \mu + (1 - t)\lambda) - \sigma > 0.
\]

Since \( \Theta_R(\mu) \) is increasing in \( R \), we see that Lemma 2.4 holds for \( N_K = R_0 \). \( \square \)

The following result will play key roles in our proof of both global and boundary second order estimates in the next two sections.

**Theorem 2.5.** Let \( K \) be a compact subset of \( C^+_\sigma \). For any \( 0 < \varepsilon < d := \text{dist}(K, \partial C^+_\sigma) \) there exist constants \( \theta_K, R_K > 0 \) such that for any \( \lambda \in \partial \Gamma^\sigma \), when \( |\lambda| \geq N_{K} \),
\[
\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta_K + \varepsilon \sum f_i(\lambda), \quad \forall \mu \in K.
\]

Furthermore, for any interval \([a, b] \subset (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)\), \( \theta_K \) and \( R_K \) can be chosen so that (2.3) holds uniformly in \( \sigma \in [a, b] \).

Proof. Let \( K^\varepsilon := \{ \mu^\varepsilon : \mu \in K \} \) where \( \mu^\varepsilon := \mu - \varepsilon \mathbf{1} \) and \( \mathbf{1} = (1, \ldots, 1) \). Then \( K^\varepsilon \subset C^+_\sigma \) and
\[
\text{dist}(K^\varepsilon, \partial C^+_\sigma) \geq d - \varepsilon > 0.
\]

As in the proof of Lemma 2.4 this implies that there exists \( R_0 > 0 \) such that
\[
\text{dist}(K^\varepsilon, T_\lambda \partial \Gamma^\sigma) \geq \frac{d - \varepsilon}{2}, \quad \forall \lambda \in \partial \Gamma^\sigma \cap \partial B_{R_0}(0).
\]

In particular, \( S_{\mu^\varepsilon}^\sigma \) lies in a compact subset \( E \) of \( \partial \Gamma^\sigma \cap \partial B_{R_0}(0) \) for all \( \mu \in K \) with \( \mu^\varepsilon \in K^\varepsilon \setminus \overline{\Gamma^\sigma} \). Equivalently,
\[
\inf_{\mu^\varepsilon \in K^\varepsilon \setminus \overline{\Gamma^\sigma}} \text{dist}_{\partial \Gamma^\sigma}(S_{\mu^\varepsilon}^\sigma, \partial \Gamma^\sigma \cap \partial B_{R_0}(0)) > 0.
\]

Let \( R > R_0 \) and \( \lambda \in \partial B_{R}(0) \cap \partial \Gamma^\sigma \). For any \( \mu^\varepsilon \in K^\varepsilon \setminus \overline{\Gamma^\sigma} \), the segment \([\mu^\varepsilon, \lambda]\) goes through \( \partial \Gamma^\sigma \) at a point \( \lambda^\varepsilon \in E \). Since \( f(\lambda) = f(\lambda^\varepsilon) = \sigma \), by the concavity of \( f \) and Lemma 2.4 we obtain when \( |\lambda| \geq N_E \),
\[
\sum f_i(\lambda)((\mu_i - \varepsilon) - \lambda_i) \geq \sum f_i(\lambda)(\lambda^\varepsilon_i - \lambda_i) \geq \omega_E > 0.
\]

Now (2.3) follows from (2.5) and Lemma 2.4 applied to \( K^\varepsilon \cap \overline{\Gamma^\sigma} \).
Finally, we note that $\theta_K$ and $R_K$ can be chosen so that they continuously depends on $\sigma$. This can be see from that fact that the hypersurfaces $\{\partial \Gamma^\sigma : \sigma \in [a, b]\}$ form a smooth foliation of the region bounded by $\partial \Gamma^a$ and $\partial \Gamma^b$, which also implies that the distance function $\text{dist}(\mu, \partial \mathcal{C}_\sigma^+)$ also depends continuously on $\sigma$ (as well as on $\mu$).

Theorem 2.5 can not be used directly in the proofs of (1.8) and (1.10) in the next two sections. So we modify it as follows.

Let $\mathcal{A}$ be the set of $n$ by $n$ symmetric matrices $A = \{A_{ij}\}$ with eigenvalues $\lambda[A] \in \Gamma$. Define the function $F$ on $\mathcal{A}$ by

$$F(A) \equiv f(\lambda[A]).$$

Throughout this paper we shall use the notation

$$F^{ij}(A) = \frac{\partial F}{\partial A_{ij}}(A), \quad F^{ijkl}(A) = \frac{\partial^2 F}{\partial A_{ij}\partial A_{kl}}(A).$$

The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$ and is positive definite by assumption (1.3), while (1.4) implies that $F$ is a concave function of $A_{ij}$ [5]. Moreover, when $A$ is diagonal so is $\{F^{ij}(A)\}$, and the following identities hold

$$F^{ij}(A)A_{ij} = \sum f_i \lambda_i,$$

$$F^{ij}(A)A_{ik}A_{kj} = \sum f_i \lambda_i^2.$$ 

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$$F^{ij}(A)A_{ij} = \sum f_i \lambda_i,$$

$$F^{ij}(A)A_{ik}A_{kj} = \sum f_i \lambda_i^2.$$

Theorem 2.5 can be rewritten as follows.

**Theorem 2.6.** Let $[a, b] \subset (\sup_{\partial \Gamma} f, \sup \Gamma f)$. For any $\sigma \in [a, b]$ and $K \subset \mathcal{A}$ such that $\lambda[K] := \{\lambda(A) : A \in K\}$ is a compact subset of $\mathcal{C}^+_{\sigma}$, there exist positive constants $\theta_K$, $R_K$ depending only on $d := \text{dist}(\lambda[K], \partial \mathcal{C}^+_{\sigma})$ and $\sup A \in K | \lambda(A)|$ (continuously), such that for any $B \in \mathcal{A}$ with $\lambda(B) \in \partial \Gamma^\sigma$, when $|\lambda(B)| \geq R_K$,

$$F^{ij}(B)(A_{ij} - B_{ij}) \geq \theta_K + \frac{d}{2} \sum F^{ii}(B).$$

Namely, (2.6) holds uniformly in $\sigma \in [a, b]$.

**Proof.** This follows immediately from Theorem 2.5 and Lemma 6.2 in [5] which is a special case of a result of M. Marcus [?]. Indeed, by Lemma 6.2 in [5] we see that

$$F^{ij}(B)A_{ij} \geq \min_{\pi} f_i(\lambda(B))\lambda_{\pi(i)}(A)$$

where the minimum is taken for all permutations $\pi$ of $\{1, \ldots, n\}$. Note also that

$$\lambda(A - \varepsilon I) = \lambda(A) - \varepsilon \mathbf{1}.$$
Theorem 2.6 now follows from Theorem 2.5.

We next present some results which are taken from [21] with minor modifications (and simplification of proof). For \( f = \sigma_k^{\frac{1}{k}} \) and \( f = (\sigma_k/\sigma_l)^{\frac{1}{k-l}} \), \( 1 \leq l < k \leq n \) they were proved earlier by Ivochkina [27] and Lin-Trudinger [33], respectively. We shall need these results when we derive the boundary estimate (1.10) in Section 4.

**Proposition 2.7.** Let \( A = \{A_{ij}\} \in \mathcal{A} \) and set \( F^{ij} = F^{ij}(A) \). There is an index \( r \) such that

\[
\sum_{l<n} F^{ij} A_{il} A_{lj} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2.
\]

**Proof.** Let \( B = \{b_{ij}\} \) be an orthogonal matrix that simultaneously diagonalizes \( \{F^{ij}\} \) and \( \{A_{ij}\} \):

\[
F^{ij} b_{li} b_{kj} = f_k \delta_{kl}, \quad A_{ij} b_{li} b_{kj} = \lambda_k \delta_{kl}.
\]

Then

\[
\sum_{l<n} F^{ij} A_{il} A_{lj} = \sum_{l<n} f_i \lambda_i^2 b_{li}^2 = \sum_{i \neq r} f_i \lambda_i^2 (1 - b_{ni}^2).
\]

Suppose for some \( r \) that \( b_{nr}^2 > \frac{1}{2} \) (otherwise we are done). Then

\[
\sum_{i \neq r} b_{ni}^2 < \frac{1}{2}.
\]

Therefore

\[
\sum_{l<n} F^{ij} A_{il} A_{lj} \geq \sum_{i \neq r} f_i \lambda_i^2 (1 - b_{ni}^2) > \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2.
\]

This proves (2.7). \[ \square \]

**Lemma 2.8.** Suppose \( f \) satisfies (1.3), (1.4) and \( \sum f_i \lambda_i \geq -K_0 \) for some constant \( K_0 \geq 0 \). Then

\[
\sum_{i \neq r} f_i \lambda_i^2 \geq \frac{1}{n+1} \sum_{i \neq r} f_i \lambda_i^2 - \frac{nK_0}{n+1} \min_{1 \leq i \leq n} \frac{1}{f_i}, \quad \text{if } \lambda_r < 0.
\]

**Proof.** Suppose \( \lambda_1 \geq \cdots \geq \lambda_n \) and \( \lambda_r < 0 \). By the concavity condition (1.4) we have \( f_n \geq f_i > 0 \) for all \( i \) and in particular \( f_n \lambda_n^2 \geq f_r \lambda_r^2 \). By (1.14),

\[
K_0 + \sum_{i \neq n} f_i \lambda_i \geq -f_n \lambda_n = f_n |\lambda_n|.
\]
It follows from Schwarz inequality that,
\[
\begin{align*}
  f_n^2 \lambda_n^2 &\leq \frac{1 + \epsilon}{\epsilon} K_0^2 + (1 + \epsilon) \sum_{i \neq n} f_i \sum_{i \neq n} f_i \lambda_i^2 \\
  &\leq \frac{1 + \epsilon}{\epsilon} K_0^2 + (1 + \epsilon)(n-1) \sum_{i \neq n} f_i \lambda_i^2 \\
  &= nK_0^2 + n f_n \sum_{i \neq n} f_i \lambda_i^2 
\end{align*}
\]
if we take \( \epsilon = \frac{1}{n-1} \). Therefore,
\[
\sum_{i \neq r} f_i \lambda_i^2 \geq \sum_{i \neq n} f_i \lambda_i^2 \geq \frac{1}{n+1} \sum_{i \neq n} f_i \lambda_i^2 + \frac{1}{n+1} f_n \lambda_n^2 - \frac{K_0^2}{f_n}
\]
completing the proof. \( \square \)

**Corollary 2.9.** Suppose \( f \) satisfies the assumptions of Lemma 2.8 For any index \( r \) and \( \epsilon > 0 \),

\[
(2.10) \quad \sum f_i |\lambda_i| \leq \epsilon \sum_{i \neq r} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i + Q(r)
\]

where \( Q(r) = f(\lambda) - f(1) \) if \( \lambda_r \geq 0 \), and
\[
Q(r) = \epsilon n K_0^2 \min_{1 \leq i \leq n} \frac{1}{f_i}, \quad \text{if} \ \lambda_r < 0.
\]

**Proof.** By the concavity of \( f \),
\[
f(1) - f(\lambda) \leq \sum f_i (1 - \lambda_i).
\]

Therefore, if \( \lambda_r \geq 0 \) then
\[
\begin{align*}
  f_r \lambda_r &\leq f(\lambda) - f(1) + \sum f_i + \sum_{\lambda_i < 0} f_i |\lambda_i| \\
  &\leq \frac{\epsilon}{2} \sum_{\lambda_i < 0} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i + f(\lambda) - f(1).
\end{align*}
\]

Suppose \( \lambda_r < 0 \). By Lemma 2.8 we have
\[
\begin{align*}
  \sum f_i |\lambda_i| &\leq \frac{\epsilon}{n+1} \sum f_i \lambda_i^2 + \frac{n+1}{4\epsilon} \sum f_i \\
  &\leq \epsilon \sum_{i \neq r} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i + \epsilon n K_0^2 \min_{1 \leq i \leq n} \frac{1}{f_i}.
\end{align*}
\]
This proves (2.10). \( \square \)
We end this section by noting the fact that $\Gamma^\sigma$ is strictly convex, and therefore (1.16) holds for $f = \sigma_k^{1/k}$, $k \geq 2$ and $f = (\sigma_k/\sigma_l)^{1/k}$, $1 \leq l < k \leq n$. Consequently Corollary 1.11 follows from Theorem 1.10.

**Lemma 2.10.** For $f = \sigma_k^{1/k}$, $k \geq 2$ or $f = (\sigma_k/\sigma_l)^{1/k}$, $1 \leq l < k \leq n$, $\partial \Gamma^\sigma = \{f = \sigma\}$ is strictly convex and, in particular, $\partial \Gamma^\sigma \cap C_\sigma^+ = \emptyset$, $\forall \sigma > 0$.

**Proof.** This was probably noticed before and may be seen in many ways; here we note that $\partial \Gamma^\sigma$ does not contain any line segment. Consider $f = (\sigma_k/\sigma_l)^{1/k}$, $0 \leq l < k \leq n$, $\partial \Gamma_k = \{f = 0\}$ since it is a polynomial. This is impossible unless $a = 0$ as $\Gamma_k$ ($k \geq 2$) does not contain whole straight lines and $f = 0$ on $\partial \Gamma_k$. 

3. **Global bounds for the second derivatives**

The goal of this section is to prove (1.8) under the hypotheses (1.3)-(1.5) and (1.7). We start with a brief explanation of our notation and basic formulas needed. Throughout the paper $\nabla$ denotes the Levi-Civita connection of $(M^n, g)$. The curvature tensor is defined by $R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z$.

Let $e_1, \ldots, e_n$ be local frames on $M^n$ and denote $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, and $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i$, etc. Define $R_{ijkl}$, $R^i_{ijkl}$ and $\Gamma^i_{kl}$ respectively by

$$R_{ijkl} = \langle R(e_k, e_l)e_j, e_i \rangle, \quad R^i_{ijkl} = g^{im}R_{mjkl}, \quad \nabla_i e_j = \Gamma^k_{ij}e_k.$$ 

For a differentiable function $v$ defined on $M^n$, we identify $\nabla v$ with the gradient of $v$, and $\nabla^2 v$ denotes the Hessian of $v$ which is given by $\nabla v_i = \nabla_i (\nabla v) - \Gamma^k_{ij} \nabla v$. Recall that $\nabla v_i = \nabla v_{ij}$ and

$$\nabla_{ijk} v - \nabla_{jik} v = R^i_{klj} \nabla_k v, \quad (3.1)$$

$$\nabla_{ijkl} v - \nabla_{ikjl} v = R^m_{ijkl} \nabla_m v, \quad (3.2)$$

$$\nabla_{ijkl} v - \nabla_{jikl} v = R^m_{klij} \nabla_m v + R^m_{lij} \nabla_{km} v. \quad (3.3)$$
From (3.2) and (3.3) we obtain
\begin{equation}
\nabla_{ijkl} v - \nabla_{klij} v = R_{mjk}^{i} \nabla_{im} v + R_{mij}^{i} \nabla_{km} v + R_{mik}^{i} \nabla_{jm} v + R_{mil}^{i} \nabla_{km} v
\end{equation}

Let \( u \in C^4(M) \) be an admissible solution of equation (1.1). Under orthonormal local frames \( e_1, \ldots, e_n \), equation (1.1) is expressed in the form
\begin{equation}
F(U_{ij}) := f(\lambda[U_{ij}]) = \psi
\end{equation}
where \( U_{ij} = \nabla_{ij} u + \chi_{ij} \). For simplicity, we shall still write equation (1.1) in the form (3.5) even if \( e_1, \ldots, e_n \) are not necessarily orthonormal, although more precisely it should be
\begin{equation}
F(\gamma^{ik} U_{kl} \gamma^{lj}) = \psi
\end{equation}
where \( \{\gamma^{ij}\} \) is the square root of \( \{g^{ij}\} \): \( \gamma^{ik} \gamma^{kj} = g^{ij} \); as long as we use covariant derivatives whenever we differentiate the equation it will make no difference.

We now begin the proof of (1.8). Let
\[ W = \max_{x \in M} \max_{\xi \in T_x M^n, |\xi| = 1} (\nabla_{\xi \xi} u + \chi(\xi, \xi)) e^n \]
where \( \eta \) is a function to be determined. Suppose \( W > 0 \) and is achieved at an interior point \( x_0 \in M \) for some unit vector \( \xi \in T_{x_0} M^n \). Choose smooth orthonormal local frames \( e_1, \ldots, e_n \) about \( x_0 \) such that \( e_1(x_0) = \xi \) and \( \{U_{ij}(x_0)\} \) is diagonal. We may also assume that \( \nabla_i e_j = 0 \) and therefore \( \Gamma_{ij}^k = 0 \) at \( x_0 \) for all \( 1 \leq i, j, k \leq n \). At the point \( x_0 \) where the function \( \log U_{11} + \eta \) (defined near \( x_0 \)) attains its maximum, we have for \( i = 1, \ldots, n \),
\begin{equation}
\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \eta = 0,
\end{equation}
\begin{equation}
\frac{\nabla_{ii} U_{11}}{U_{11}} - \left( \frac{\nabla_i U_{11}}{U_{11}} \right)^2 + \nabla_{ii} \eta \leq 0.
\end{equation}

Here we wish to add some explanations which might be helpful to the reader. First we note that \( U_{1j}(x_0) = 0 \) for \( j \geq 2 \) so \( \{U_{ij}(x_0)\} \) can be diagonalized. To see this let \( e^\theta = e_1 \cos \theta + e_j \sin \theta \). Then
\[ U_{e^\theta e^\theta}(x_0) = U_{11} \cos^2 \theta + 2U_{1j} \sin \theta \cos \theta + U_{jj} \sin^2 \theta \]
has a maximum at \( \theta = 0 \). Therefore,
\[ \frac{d}{d\theta} U_{e^\theta e^\theta}(x_0) \bigg|_{\theta=0} = 0. \]
This gives $U_{ij}(x_0) = 0$.

Next, at $x_0$ we have

\[(3.8) \quad \nabla_i (U_{11}) = \nabla_i U_{11},\]

that is $e_i(U_{11}) = \nabla_i U_{11} \equiv \nabla^3 u(e_1, e_1, e_i) + \nabla \chi(e_1, e_1, e_i)$, and

\[(3.9) \quad \nabla_{ij}(U_{11}) = \nabla_{ij} U_{11}.\]

One can see (3.8) immediately if we assume $\Gamma^k_{ij} = 0$ at $x_0$ for all $1 \leq i, j, k \leq n$. In general, we have

$$\nabla_i(U_{11}) = \nabla_i U_{11} + 2\Gamma^k_{ij} U_{1k} = \nabla_i U_{11} + 2\Gamma^1_{i1} U_{11}$$
as $U_{1k}(x_0) = 0$. On the other hand, since $e_1, \ldots, e_n$ are orthonormal,

$$g(\nabla_k e_i, e_j) = g(e_i, \nabla_k e_j) = 0$$

and

$$g(\nabla_i e_1, \nabla_j e_1) + g(e_1, \nabla_i \nabla_j e_1) = 0.$$ 
Thus

\[(3.10) \quad \Gamma^j_{ki} + \Gamma^i_{kj} = 0\]

and

$$\Gamma^k_{i1} \Gamma^j_{kj} + \nabla_i (\Gamma^1_{j1}) + \Gamma^k_{j1} \Gamma^1_{ik} = 0.$$ 
This gives $\Gamma^1_{i1} = 0$ and $\nabla_i (\Gamma^1_{j1}) = 0$. So we have (3.8).

For (3.9) we calculate directly,

$$\nabla_{ij}(U_{11}) = \nabla_i (\nabla_j(U_{11})) - \Gamma^k_{ij} \nabla_k(U_{11})$$

$$= \nabla_i (\nabla_j U_{11} + 2\Gamma^k_{j1} U_{1k}) - \Gamma^k_{ij} \nabla_k U_{11}$$

$$= \nabla_i U_{11} + \Gamma^k_{ij} \nabla_k U_{11} + 2\Gamma^k_{i1} \nabla_j U_{1k} + 2\nabla_i (\Gamma^k_{j1}) U_{1k}$$

$$+ 2\Gamma^k_{j1} \nabla_i U_{1k} + 2\Gamma^k_{i1} \Gamma^1_{j1} U_{1k} - \Gamma^k_{ij} \nabla_k U_{11}$$

$$= \nabla_i U_{11} + 2\Gamma^k_{i1} \nabla_j U_{1k} + 2\nabla_i (\Gamma^1_{j1}) U_{1k} + 2\Gamma^k_{i1} \Gamma^1_{j1} U_{1k} - 2\Gamma^k_{ii} \Gamma^k_{j1} U_{11}$$

by (3.10) and $\nabla_i (\Gamma^1_{j1}) = 0$. Therefore we have (3.9) if $\Gamma^k_{ij} = 0$ at $x_0$.

We now continue our proof of (1.8). Differentiating equation (3.5) twice, we obtain at $x_0$,

\[(3.11) \quad F^{ij} \nabla_k U_{ij} = \nabla_k \psi, \quad \text{for all } k,\]

\[(3.12) \quad F^{ii} \nabla_{11} U_{ii} + \sum F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} = \nabla_{11} \psi.\]
Here and throughout rest of the paper, $F^{ij} = F^{ij} \{U_{ij}\}$. By (3.4),
\[
F^{ii} \nabla_{ii} U_{11} \geq F^{ii} \nabla_{11} U_{ii} + 2F^{ii} R_{1i1i}(\nabla_{11}u - \nabla_{ii}u) - C \sum F^{ii}
\]
(3.13)
\[
\geq F^{ii} \nabla_{11} U_{ii} - C(1 + U_{11}) \sum F^{ii}.
\]
Here we note that $C$ depends on the gradient bound $|\nabla u|_{C^0(M)}$. From (3.7), (3.12) and (3.13) we derive
\[
(3.14)
U_{11} F^{ii} \nabla_{ii} \eta \leq E - \nabla_{11} \psi + C(1 + U_{11}) \sum F^{ii}
\]
where
\[
E \equiv F^{ij,kl} \nabla_{1i} U_{ij} \nabla_{1k} U_{kl} + \frac{1}{U_{11}} F^{ii}(\nabla_i U_{11})^2.
\]
To estimate $E$ let $0 < s < 1$ (to be chosen) and
\[
J = \{i : U_{ii} \leq -sU_{11}\}, \quad K = \{i > 1 : U_{ii} > -sU_{11}\}.
\]
It was shown by Andrews [1] and Gerhardt [9] that
\[
-F^{ij,kl} \nabla_{1i} U_{ij} \nabla_{1k} U_{kl} \geq \sum_{i \neq j} \frac{F^{ii} - F^{jj}}{U_{jj} - U_{ii}} (\nabla_1 U_{ij})^2.
\]
Therefore,
\[
-F^{ij,kl} \nabla_{1i} U_{ij} \nabla_{1k} U_{kl} \geq 2 \sum_{i \geq 2} \frac{F^{ii} - F^{11}}{U_{11} - U_{ii}} (\nabla_1 U_{ii})^2
\]
\[
\geq 2 \sum_{i \in K} \frac{F^{ii} - F^{11}}{U_{11} - U_{ii}} (\nabla_1 U_{ii})^2
\]
(3.15)
\[
\geq 2 \frac{1}{(1 + s)U_{11}} \sum_{i \in K} (F^{ii} - F^{11})(\nabla_1 U_{ii})^2
\]
\[
\geq 2 \frac{(1 - s)}{(1 + s)U_{11}} \sum_{i \in K} (F^{ii} - F^{11})[(\nabla_i U_{11})^2 - C/s].
\]
We now fix $s \leq 1/3$ and hence
\[
\frac{2(1 - s)}{1 + s} \geq 1.
\]
From (3.15) and (3.6) it follows that
\[
E \leq \frac{1}{U_{11}} \sum_{i \in J} F^{ii}(\nabla_i U_{11})^2 + \frac{C}{U_{11}} \sum_{i \in K} F^{ii} + \frac{CF^{11}}{U_{11}} \sum_{i \notin J} (\nabla_i U_{11})^2
\]
(3.16)
\[
\leq U_{11} \sum_{i \in J} F^{ii}(\nabla_\eta)^2 + \frac{C}{U_{11}} \sum_{i \in J} F^{ii} + C U_{11} F^{11} \sum_{i \notin J} (\nabla_\eta)^2.
\]
Let 
\[ \eta = \phi(|\nabla u|^2) + a(u - u) \]
where \( \phi \) is a positive function, \( \phi' > 0 \), and \( a \) is a positive constant. We calculate
\[ \nabla_i \eta = 2\phi' \nabla_k u \nabla_{ik} u + a \nabla_i (u - u) \]
\[ = 2\phi' (U_{ii} \nabla_i u - \chi_{ik} \nabla_k u) + a \nabla_i (u - u), \]
\[ \nabla_{ii} \eta = 2\phi' (\nabla_{ik} u \nabla_{ik} u + \nabla_k u \nabla_{iik} u) + 2\phi'' (\nabla_k u \nabla_{ik} u)^2 + a \nabla_{ii} (u - u). \]

Therefore,
\[ \sum_{i \in J} F_{ii}(\nabla_i \eta)^2 \leq 8(\phi')^2 \sum_{i \in J} F_{ii}(\nabla_k u \nabla_{ik} u)^2 + C a^2 \sum_{i \in J} F_{ii}, \]
(3.17)
\[ \sum_{i \notin J} (\nabla_i \eta)^2 \leq C(\phi')^2 U_{11}^2 + C(\phi')^2 + C a^2 \]
and by (3.11),
\[ F_{ii} \nabla_{ii} \eta \geq \phi' F_{ii} U_{ii}^2 + 2\phi'' F_{ii} (\nabla_k u \nabla_{ik} u)^2 \]
\[ + a F_{ii} \nabla_{ii} (u - u) - C \phi' (1 + \sum F_{ii}). \]
(3.19)

Let \( \phi(t) = b(1 + t)^2 \); we may assume \( \phi'' - 4(\phi')^2 = 2b(1 - 8\phi) \geq 0 \) in any fixed interval \([0, C_1]\) by requiring \( b > 0 \) sufficiently small. Combining (3.14), (3.16), (3.17), (3.18) and (3.19), we obtain
\[ \phi' F_{ii} U_{ii}^2 + a F_{ii} \nabla_{ii} (u - u) \leq C a^2 \sum_{i \in J} F_{ii} + C((\phi')^2 U_{11}^2 + a^2) F^{11} \]
\[ + \frac{\nabla_{11} \psi}{U_{11}} + C \left( 1 + \sum F_{ii} \right). \]
(3.20)

Suppose \( U_{11}(x_0) > R \) sufficiently large and apply Theorem 2.6 to \( A = \{ \nabla_{ij} u + \chi_{ij} \} \) and \( B = \{ U_{ij} \} \) at \( x_0 \). We see that
\[ F_{ii} \nabla_{ii} (u - u) = F_{ii}[(\nabla_{ii} u + \chi_{ii}) - U_{ii}] \geq \theta \left( 1 + \sum F_{ii} \right). \]
Plug this into (3.20) and fix \( a \) sufficiently large. We derive
\[ \phi' F_{ii} U_{ii}^2 \leq C a^2 \sum_{i \in J} F_{ii} + C((\phi')^2 U_{11}^2 + a^2) F^{11}. \]
(3.21)

Note that
\[ F_{ii} U_{ii}^2 \geq F^{11} U_{11}^2 + \sum_{i \in J} F_{ii} U_{ii}^2 \geq F^{11} U_{11}^2 + s^2 U_{11} U_{ii} \sum_{i \in J} F_{ii}. \]
Fixing $b$ sufficiently small we obtain from (3.21) a bound $U_{11} \leq Ca/\sqrt{b}$. This implies (1.8), and (1.9) when $M$ is closed.

4. Boundary estimates

In this section we establish the boundary estimate (1.10) under the assumptions of Theorem 1.5. Throughout this section we assume the function $\varphi \in C^4(\partial M)$ is extended to a $C^4$ function on $\bar{M}$, still denoted $\varphi$.

For a point $x_0$ on $\partial M$, we shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around $x_0$ such that when restricted to $\partial M$, $e_n$ is normal to $\partial M$.

Let $\rho(x)$ denote the distance from $x$ to $x_0$, 
$$
\rho(x) \equiv \text{dist}_{M^*}(x, x_0),
$$
and $M_\delta = \{x \in M : \rho(x) < \delta\}$. Since $\partial M$ is smooth we may assume the distance function to $\partial M$
$$
d(x) \equiv \text{dist}(x, \partial M)
$$
is smooth in $M_{\delta_0}$ for fixed $\delta_0 > 0$ sufficiently small (depending only on the curvature of $M$ and the principal curvatures of $\partial M$.) Since $\nabla_{ij}\rho^2(x_0) = 2\delta_{ij}$, we may assume $\rho$ is smooth in $M_{\delta_0}$ and
$$
\{\delta_{ij}\} \leq \{\nabla_{ij}\rho^2\} \leq 3\{\delta_{ij}\} \quad \text{in} \quad M_{\delta_0}.
$$

The following lemma which crucially depends on Theorem 2.6 plays key roles in our boundary estimates.

**Lemma 4.1.** There exist some uniform positive constants $t, \delta, \varepsilon$ sufficiently small and $N$ sufficiently large such that the function

$$
v = (u - \bar{u}) + td - \frac{Nd^2}{2}
$$
satisfies $v \geq 0$ on $\bar{M}_\delta$ and

$$
F^{ij}\nabla_{ij}v \leq -\varepsilon \left(1 + \sum F^{ii}\right) \quad \text{in} \quad M_\delta.
$$

**Proof.** We note that to ensure $v \geq 0$ in $\bar{M}_\delta$ we may require $\delta \leq 2t/N$ after $t, N$ being fixed. Obviously,

$$
F^{ij}\nabla_{ij}v = F^{ij}\nabla_{ij}(u - \bar{u}) + (t - Nd)F^{ij}\nabla_{ij}d - NF^{ij}\nabla_{i}d\nabla_{j}d
$$
$$
\leq C_1(t + Nd)\sum F^{ii} + F^{ij}\nabla_{ij}(u - \bar{u}) - NF^{ij}\nabla_{i}d\nabla_{j}d.
$$
Fix $\varepsilon > 0$ sufficiently small and $R \geq R_A$ so that Theorem 2.6 holds for $A = \{\nabla_{ij}u + \chi_{ij}\}$ and $B = \{U_{ij}\}$ at every point in $\bar{M}_\delta$. Let $\lambda = \lambda[\{U_{ij}\}]$ be the eigenvalues of $\{U_{ij}\}$. At a fixed point in $M_\delta$ we consider two cases: (a) $|\lambda| \leq R$; and (b) $|\lambda| > R$.

In case (a) there are uniform bounds (depending on $R$) $0 < c_1 \leq \{F_{ij}\} \leq C_1$ and therefore $F_{ij}\nabla_i\nabla_j d \geq c_1$ since $|\nabla d| \equiv 1$. We may fix $N$ large enough so that (4.3) holds for any $t, \varepsilon \in (0, 1]$, as long as $\delta$ is sufficiently small.

In case (b) by Theorem 2.6 and (4.4) we may further require $t$ and $\delta$ so that (4.3) holds for some different (smaller) $\varepsilon > 0$. □

We now start the proof of (1.10). Consider a point $x_0 \in \partial M$. Since $u - u = 0$ on $\partial M$ we have

$$\nabla_{ij}(u - u) = -\nabla_n(u - u)\Pi(e_\alpha, e_\beta), \ \forall \ 1 \leq \alpha, \beta < n \text{ on } \partial M$$

where $\Pi$ denotes the second fundamental form of $\partial M$. Therefore,

$$|\nabla_{\alpha\beta}u| \leq C, \ \forall \ 1 \leq \alpha, \beta < n \text{ on } \partial M.$$

To estimate the mixed tangential-normal and pure normal second derivatives we note the following formula

$$\nabla_{ij}(\nabla_k u) = \nabla_{ijk} u + \Gamma^l_{ik}\nabla_jl u + \Gamma^l_{jk}\nabla_il u + \nabla_{ij}\epsilon_k u.$$ 

By (3.11), therefore,

$$|F_{ij}\nabla_{ij}\nabla_k(u - \varphi)| \leq 2F_{ij}\Gamma^l_{ik}\nabla_jl u + C\left(1 + \sum F_{ii}\right)$$

$$\leq C\left(1 + \sum f_i|\lambda_i| + \sum f_i\right).$$

Let

$$\Psi = A_1v + A_2\rho^2 - A_3\sum_{\beta<n}|\nabla_\beta(u - \varphi)|^2.$$

By (4.7) we have

$$F_{ij}\nabla_{ij}|\nabla_\beta(u - \varphi)|^2 = 2F_{ij}\nabla_\beta(u - \varphi)\nabla_{ij}\nabla_\beta(u - \varphi)$$

$$+ 2F_{ij}\nabla_i\nabla_\beta(u - \varphi)\nabla_j\nabla_\beta(u - \varphi)$$

$$\geq F_{ij}U_{i\beta}U_{j\beta} - C\left(1 + \sum f_i|\lambda_i| + \sum f_i\right).$$
For fixed $1 \leq \alpha < n$, by Lemma 4.1, Proposition 2.7 and Corollary 2.9 we see that
\begin{equation}
F^{ij} \nabla_{ij}(\Psi \pm \nabla_{\alpha}(u - \varphi)) \leq 0, \text{ in } M_{\delta}
\end{equation}
and $\Psi \pm \nabla_{\alpha}(u - \varphi) \geq 0$ on $\partial M_{\delta}$ when $A_1 \gg A_2 \gg A_3 \gg 1$. By the maximum principle we derive $\Psi \pm \nabla_{\alpha}(u - \varphi) \geq 0$ in $M_{\delta}$ and therefore
\begin{equation}
|\nabla_{n\alpha}u(x_0)| \leq \nabla_{n}\Psi(x_0) + |\nabla_{n\alpha}\varphi(x_0)| \leq C, \forall \alpha < n.
\end{equation}

It remains to derive
\begin{equation}
\nabla_{nn}u(x_0) \leq C.
\end{equation}
We show this by proving that there are uniform constants $c_0, R_0$ such that for all $R > R_0$, $(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) \in \Gamma$ and
\begin{align*}
f(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) &\geq \psi(x_0) + c_0
\end{align*}
where $\lambda'[\{U_{\alpha\beta}\}] = (\lambda'_1, \ldots, \lambda'_{n-1})$ denotes the eigenvalues of the $(n-1) \times (n-1)$ matrix $\{U_{\alpha\beta}\}$ $(1 \leq \alpha, \beta \leq n-1)$. Suppose we have found such $c_0$ and $R_0$. By Lemma 1.2 of [5], from estimates (4.6) and (4.11) we can find $R_1 \geq R_0$ such that if $U_{nn}(x_0) > R_1$,
\begin{align*}
f(\lambda'[\{U_{ij}(x_0)\}]) &\geq f(\lambda'[\{U_{\alpha\beta}(x_0)\}], U_{nn}(x_0)) - \frac{c_0}{2}.
\end{align*}
By equation (1.1) this gives a desired bound $U_{nn}(x_0) \leq R_1$ for otherwise, we would have a contradiction:
\begin{align*}
f(\lambda'[\{U_{ij}(x_0)\}]) &\geq \psi(x_0) + \frac{c_0}{2}.
\end{align*}

For $R > 0$ and a symmetric $(n-1)^2$ matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}(x_0)\}], R) \in \Gamma$, define
\begin{align*}
\tilde{F}[r_{\alpha\beta}] &\equiv f(\lambda'[\{r_{\alpha\beta}\}], R).
\end{align*}
Following an idea of Trudinger [37] we consider
\begin{align*}
m_R &\equiv \min_{x_0 \in \partial M} \tilde{F}[U_{\alpha\beta}(x_0)] - \psi(x_0).
\end{align*}
Note that $\tilde{F}$ is concave and $m_R$ is increasing in $R$ by (1.3), and that
\begin{align*}
c_R &\equiv \inf_{\partial M}(\tilde{F}[U_{\alpha\beta}] - F[U_{ij}]) > 0
\end{align*}
when $R$ is sufficiently large.
We wish to show \( m_R > 0 \) for \( R \) sufficiently large. Suppose \( m_R \) is achieved at a point \( x_0 \in \partial M \). Choose local orthonormal frames around \( x_0 \) as before and let

\[
\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r^{\alpha\beta}}[U_{\alpha\beta}(x_0)].
\]

Since \( \tilde{F} \) is concave, for any symmetric matrix \( \{r_{\alpha\beta}\} \) with \( (\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma \),

\[
(4.13) \quad \tilde{F}_0^{\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0)) \geq \tilde{F}[r_{\alpha\beta}] - \tilde{F}[U_{\alpha\beta}(x_0)].
\]

In particular,

\[
(4.14) \quad \tilde{F}_0^{\alpha\beta} U_{\alpha\beta} - \psi - \tilde{F}_0^{\alpha\beta} U_{\alpha\beta}(x_0) + \psi(x_0) \geq \tilde{F}[U_{\alpha\beta}] - \psi - m_R \geq 0 \quad \text{on} \quad \partial M.
\]

By (4.5) we have on \( \partial M \),

\[
(4.15) \quad U_{\alpha\beta} = U_{\alpha\beta} - \nabla_n(u - \tilde{u})\sigma_{\alpha\beta}
\]

where \( \sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle \); note that \( \sigma_{\alpha\beta} = \Pi(e_\alpha, e_\beta) \) on \( \partial M \). It follows that

\[
\nabla_n(u - \tilde{u})\tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) = \tilde{F}_0^{\alpha\beta}(U_{\alpha\beta}(x_0) - U_{\alpha\beta}(x_0))
\]

\[
\geq \tilde{F}[U_{\alpha\beta}(x_0)] - \tilde{F}[U_{\alpha\beta}(x_0)]
\]

\[
= \tilde{F}[U_{\alpha\beta}(x_0)] - \psi(x_0) - m_R \geq \psi(x_0) - \psi(x_0) - m_R \geq c_R - m_R.
\]

Consequently, if

\[
\nabla_n(u - \tilde{u})(x_0)\tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) \leq \frac{c_R}{2}
\]

then \( m_R \geq c_R/2 \) and we are done.

Suppose now that

\[
\nabla_n(u - \tilde{u})(x_0)\tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) > \frac{c_R}{2}
\]

and let \( \eta \equiv \tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta} \). Note that

\[
(4.16) \quad \eta(x_0) \geq c_R/2\nabla_n(u - \tilde{u})(x_0) \geq 2\epsilon_1 c_R
\]

for some uniform \( \epsilon_1 > 0 \) independent of \( R \). We may assume \( \eta \geq \epsilon_1 c_R \) on \( \bar{M}_\delta \) by requiring \( \delta \) small. Define in \( M_\delta \),

\[
\Phi = -\nabla_n(u - \varphi) + \frac{1}{\eta} \tilde{F}_0^{\alpha\beta}(\nabla_\alpha\beta \varphi + \chi_{\alpha\beta} - U_{\alpha\beta}(x_0)) - \frac{\psi - \psi(x_0)}{\eta}
\]

\[
\equiv -\nabla_n(u - \varphi) + Q.
\]

We have \( \Phi(x_0) = 0 \) and \( \Phi \geq 0 \) on \( \partial M \) near \( x_0 \) by (4.14) since

\[
\nabla_{\alpha\beta}u = \nabla_{\alpha\beta} \varphi - \nabla_n(u - \varphi)\sigma_{\alpha\beta} \quad \text{on} \quad \partial M,
\]
while by (4.7),
\[
F^{ij} \nabla_{ij} \Phi \leq -F^{ij} \nabla_{ij} \nabla_n u + C \sum F^{ii} \leq C \left(1 + \sum f_i |\lambda_i| + \sum f_i \right). \tag{4.17}
\]

Consider the function $\Psi$ defined in (4.8). Applying Lemma 4.1, Proposition 2.7 and Corollary 2.9 as before for $A_1 \gg A_2 \gg A_3 \gg 1$ we derive $\Psi + \Phi \geq 0$ on $\partial M_\delta$ and (4.18)
\[
F^{ij} \nabla_{ij} (\Psi + \Phi) \leq 0 \text{ in } M_\delta.
\]
By the maximum principle, $\Psi + \Phi \geq 0$ in $M_\delta$. Thus $\Phi_n(x_0) \geq -\nabla_n \Psi(x_0) \geq -C$. This gives $\nabla_{nn} u(x_0) \leq C$.

So we have an a priori upper bound for all eigenvalues of $\{U_{ij}(x_0)\}$. Consequently, $\lambda [\{U_{ij}(x_0)\}]$ is contained in a compact subset of $\Gamma$ by (1.5), and therefore
\[
m_R = \tilde{F}[U_{\alpha\beta}(x_0)] - \psi(x_0) > 0
\]
when $R$ is sufficiently large. This completes the proof of (1.10).

5. The Gradient Estimates and Proof of Theorem 1.10

By Theorems 1.1-1.5 and Evans-Krylov theorem, one only needs to derive a priori $C^1$ estimates in order to prove Theorem 1.10 using the continuity method. It seems an interesting question whether one can prove gradient estimates under assumption (1.7). We wish to come back to the problem in future work. Here we only list some conditions for gradient estimates that were more or less known to Li [31] and Urbas [39].

**Proposition 5.1.** Let $u \in C^2(\bar{M})$ be an admissible solution of equation (1.1) where $\psi \in C^1(\bar{M})$. Suppose $f$ satisfies (1.3)-(1.5). Then
\[
\max_M |\nabla u| \leq C \left(1 + \max_{\partial M} |\nabla u| \right) \tag{5.1}
\]
where $C$ depends on $|u|_{C^0(\bar{M})}$, under any of the following additional assumptions: (i) $\Gamma = \Gamma_n$. (ii) (1.7) and that $(M, g)$ has nonnegative sectional curvature; (iii) (1.14) and (1.17) for $|\lambda|$ sufficiently large.

**Proof.** Case (i): $\Gamma = \Gamma_n$. For fixed $A > 0$ suppose $Au + |\nabla u|^2$ has a maximum at an interior point $x_0 \in M$. Then $A \nabla_i u + 2 \nabla_k u \nabla_{ki} u = \nabla_k u (A \delta_{ki} + \nabla_{ki} u) = 0$ at $x_0$ for all $1 \leq i \leq n$. This implies $\nabla u(x_0) = 0$ when $A$ is sufficiently large. Therefore,
\[
\sup_M |\nabla u|^2 \leq A \left( \sup_M u - \inf_M u \right) + \sup_{\partial M} |\nabla u|^2.
\]
Case (iii') was proved by Urbas [39] under the additional assumption

\[ \sum f_i(\lambda) \geq \delta_\sigma, \forall \lambda \in \partial \Gamma^\sigma, \sigma > \sup_{\partial \Gamma} f, \]

which is in fact implied by (1.14). Indeed, by the concavity of \( f \) and (1.14),

\[ A \sum f_\lambda(\lambda) \geq \sum f_\lambda(\lambda) \lambda_i + f(A1) - f(\lambda) \geq f(A1) - \sigma \]

for any \( \lambda \in \Gamma, f(\lambda) = \sigma \). Fixing \( A \) sufficiently large gives (5.2).

Case (ii') is a slight improvement of the gradient estimates derived by Li [31]. So we only outline a modification of the proof in [31].

Suppose \( |\nabla u|^2 e^\phi \) achieves a maximum at an interior point \( x_0 \in M \). Then at \( x_0 \),

\[
\frac{2\nabla_k u \nabla_{ik} u}{|\nabla u|^2} + \nabla_i \phi = 0, \\
2F^{ij}(\nabla_k u \nabla_{jk} u + \nabla_{ik} u \nabla_{jk} u) + |\nabla u|^2 F^{ij}(\nabla_{ij} \phi - \nabla_i \phi \nabla_j \phi) \leq 0.
\]

Following [31] we use the nonnegative sectional curvature condition to derive

\[ |\nabla u| F^{ij}(\nabla_{ij} \phi - \nabla_i \phi \nabla_j \phi) \leq C \sum F^{ii} + C \]

Now let \( \phi = A(1 + u - u + \sup(u - u)) \) and fix \( A > 0 \) sufficiently small. From (5.3),

\[ 2A F^{ij} \nabla_{ij}(u - u) + 2A(1 - 2A) F^{ij} \nabla_i (u - u) \nabla_j (u - u) \leq \frac{C}{|\nabla u|} \sum F^{ii} + \frac{C}{|\nabla u|}.
\]

By (1.7) and Theorem 2.6 we derive a bound \( |\nabla u(x_0)| \leq C \) if \( |\lambda[\nabla^2 u + \chi](x_0)| \geq R \) for \( R \) sufficiently large.

Suppose \( |\lambda[\nabla^2 u + \chi](x_0)| \leq R \). By (1.3) and (1.5) there exists \( C_1 > 0 \) depending on \( R \) such that at \( x_0 \),

\[ \frac{g^{-1}}{C_1} \leq \{F^{ij}\} \leq C_1 g^{-1}. \]

Then

\[ 2A(1 - 2A)C_1^{-1}|\nabla(u - u)|^2 \leq \frac{C}{|\nabla u|}. \]

We derive a bound for \( |\nabla u(x_0)| \) again. \( \square \)

By the maximum principle we have \( u \leq u \leq h \) where \( h \in C^2(\bar{M}) \) is the solution of \( \Delta h + \text{tr}\chi = 0 \) in \( \bar{M} \) with \( h = \varphi \) on \( \partial M \). This gives bounds for \( |u|_{C^0(\bar{M})} \) and \( |\nabla u| \) on \( \partial M \). The proof of Theorem 1.10 using the continuity method is standard and therefore omitted here.
References

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