Invent. math. 83, 153–166 (1986)

Inventiones mathematicae

Minimal immersions of surfaces by the first Eigenfunctions and conformal area

Sebastián Montiel and Antonio Ros

Departamento de Geometria, Universidad de Granada, E-18071 Granada, Spain

Introduction

Let $\psi: M \to S^n$ be a minimal immersion of a compact surface into a unit sphere. Then, the linear functions of ψ are eigenfunctions for the Laplacian of M corresponding to the eigenvalue $\lambda = 2$. The main purpose of this paper is to study those minimal surfaces for which 2 is exactly the first non-zero eigenvalue of its Laplacian. This kind of immersions have a peculiar behaviour among all compact minimal surfaces of the sphere and they appear naturally when one considers different geometric problems, as Li and Yau have shown in [6]. The methods that we use in this paper are based, for the most part, on [6].

It is known that the only metric on a 2-dimensional sphere admitting a minimal immersion into S^n by the first eigenfunctions is the standard one (this follows, for example, from the fact that the multiplicity of the first eigenvalue for such a metric is at most three, see the Cheng work [3]). Our first result shows that it is possible to extend this property for an arbitrary compact surface, in the following way:

> "For each conformal structure on a compact surface, there exists at most one metric admitting a minimal immersion into a unit sphere by the first eigenfunctions".

As a consequence, the class consisting of such immersions is not too big. This result enables us to characterize the equalities in some inequalities obtained by Li and Yau which relate the conformal area of a Riemannian surface, the first non-zero eigenvalue of its Laplacian and the total mean curvature for a surface in the Euclidean space.

Since the real projective plane has only one conformal structure, the only metric on RP^2 admitting a minimal immersion into a sphere by the first eigenfunctions is the standard one. Thus, the metrics on S^2 or RP^2 which have this type of immersions are completely classified. Evidently, we are interested in extending this classification for other compact surfaces. Besides S^2 or RP^2, the torus has the simplest family of conformal structures. The square and equilateral flat tori are the only known examples of Riemannian tori admitting a minimal immersion into a sphere by the first eigenfunctions (these immersions lie in S^3 and S^5 respectively). For this surface we obtain the following partial classification result:

"The only minimal torus immersed into S^3 by the first eigenfunctions is the Clifford torus".

This result gives us a relation between the Lawson conjecture, which asserts that the only torus minimally embedded into S^3 is the Clifford torus, and the Yau conjecture, which says that each minimal embedding of a compact surface into S^3 is by the first eigenfunctions. Also, note that, if the Yau conjecture is true, it follows from our first result that two compact surfaces minimally embedded into S^3 are isometric provided that they are conformally equivalent.

Now, it seems natural to study, for a fixed conformal structure on a compact surface, the problem of existence of metrics admitting this type of immersions. We will show that, in general, one cannot assure such existence. More concretly, we will prove the following fact:

"There exists conformal structures on a torus for which there are no metrics admitting a minimal immersion into any sphere by the first eigenfunctions".

This follows from certain results having their own interest that we will point out in the following. Li and Yau estimate the conformal area of a torus in terms of the area and the first eigenvalue of the only flat metric existing for each conformal structure. We improve their bound for the conformal area and this enables us to enlarge the family of conformal structures on a torus for which the Willmore conjecture is satisfied. Also, we compute the conformal area for a class of rectangular tori and, finally, we prove that, for these conformal structures, there are no metrics which admit minimal immersions by the first eigenfunctions. Since the conformal area of these tori is less than $2\pi^2$, it follows that the estimate of the total mean curvature of a torus in R^n in terms of its conformal area obtained in [6] is not sufficient to verify the Willmore conjecture in the general case.

1. Preliminaries

Let S^n be the n-dimensional unit sphere, D^{n+1} the open unit ball bounded by S^n in R^{n+1} and G the conformal group of S^n. For each point $g \in D^{n+1}$, we consider the map, also denoted by g, $g: S^n \to S^n$ given by

$$g(p) = \frac{p + (\mu\langle p, g\rangle + \lambda) g}{\lambda(\langle p, g\rangle + 1)} \tag{1.1}$$

for all $p \in S^n$, where $\lambda = (1-|g|^2)^{-\frac{1}{2}}$, $\mu = (\lambda - 1)|g|^{-2}$ and $\langle\ ,\ \rangle$ denotes the usual inner product on R^{n+1}. Then g is a conformal transformation of S^n which can be extended to an isometry of D^{n+1} endowed with the hyperbolic metric, that carries the origin of D^{n+1} on the point g. Moreover, each transformation of G can be expressed by $T \circ g$, where T is an orthogonal transformation of S^n and g is given by (1.1) for some $g \in D^{n+1}$.

The differential map g_* of g is given by

$$g_*(v)=\lambda^{-2}(\langle p,g\rangle+1)^{-2}\{\lambda(\langle p,g\rangle+1)v-\lambda\langle v,g\rangle p+\langle v,g\rangle(1-\lambda)|g|^{-2}g\}, \tag{1.2}$$

where v is a tangent vector to S^n at p. So, for such two vectors v and w we have

$$\langle g_*(v),g_*(w)\rangle=\frac{1-|g|^2}{(\langle p,g\rangle+1)^2}\langle v,w\rangle. \tag{1.3}$$

Let M be a surface. We consider two conformal metrics ds^2 and ds_1^2 on M related by $ds_1^2=2F\,ds^2$, where F is a regular positive function. If Δ and dM (resp. Δ_1 and dM_1) are the Laplacian and the canonical measure on M corresponding to the metric ds^2 (resp. ds_1^2), one has the relations $\Delta_1=(2F)^{-1}\Delta$ and $dM_1=2F\,dM$. Now, if M is compact with metric ds^2, for each branched conformal immersion $\psi\colon M\to S^n$ we can consider the area function $A\colon G\to R$ which maps a conformal transformation h of S^n on the area induced from the immersion $h\circ\psi$. We will denote it by $A(h\circ\psi)$ or simply $A(h)$. Since the area function is invariant under orthogonal transformations of S^n, we can ourselves restrict to conformal transformations of the type (1.1). So, the area function can be defined on the unit ball, $A\colon D^{n+1}\to R$. From (1.3), we obtain for each $g\in D^{n+1}$

$$A(g)=\tfrac{1}{2}\int_M\frac{1-|g|^2}{(\langle\psi,g\rangle+1)^2}|\nabla\psi|^2\,dM. \tag{1.4}$$

Note that, if the immersion ψ is isometric, then $|\nabla\psi|^2=2$.

From the first variational formula for the area function or by a direct computation, we have that the origin of D^{n+1} is a critical point for A if and only if

$$\int_M\psi|\nabla\psi|^2\,dM=0. \tag{1.5}$$

In the same way, the second variational formula at the origin, in the direction of $v\in R^{n+1}$, is given by

$$\int_M\{3\langle\psi,v\rangle^2-|v|^2\}|\nabla\psi|^2\,dM. \tag{1.6}$$

As consequence of (1.6), one can see that the function $A\colon D^{n+1}\to R$ satisfies

$$\Delta A=\tfrac{1}{2}(2-n)A, \tag{1.7}$$

where Δ is the Laplacian corresponding to the hyperbolic metric on D^{n+1} with constant curvature -1. The behaviour of A at the infinity, that is, when $g\to\theta\in S^n$, has been studied by Li and Yau. In fact, we have $A(g)\to4k\pi$ where $k=0,1,\dots$ is the number of points of M that are mapped on $-\theta$ by the immersion ψ.

Suppose that $\psi\colon M\to S^n$ is an isometric immersion. Let $\bar{H}$ and H be the mean curvature vectors of M in S^n and R^{n+1} respectively. Then

$$\int_M|H|^2\,dM=\int_M|\bar{H}|^2\,dM+A(M). \tag{1.8}$$

It is well-known that the left side in the above equality is invariant under conformal transformations of G. It follows that, if ψ is a minimal immersion, the corresponding area function attains its maximum at the origin of D^{n+1}.

Let M be a compact surface endowed with a fixed conformal structure. The conformal n-area of M, that we will denote by $A_c(M,n)$ is given by

$$A_c(M,n)=\inf_{\psi}\sup_{g\in D^{n+1}} A(g\circ\psi)$$

where ψ runs over all non-degenerate conformal mappings of M into S^n. The conformal area $A_c(M)$ is defined by $A_c(M)=\inf_n A_c(M,n)$, $n\geqq 2$.

Now, we will point out some other results obtained by Li and Yau in [6]. Let M be a compact surface with metric ds^2 and canonical measure dM:

(1.9) "If $\psi: M\to S^n$ is a conformal immersion, then there exists $g\in D^{n+1}$ such that $\int_M g\circ\psi\, dM=0$".

If λ_1 denotes the first non-zero eigenvalue of the Laplacian, then

$$\lambda_1(M)\,A(M)\leqq 2A_c(M,n). \tag{1.10}$$

Moreover, equality holds if and only if M admits a minimal immersion into a n-dimensional sphere which is given by a subspace of the first eigenspace. In this case $2A_c(M)=\lambda_1(M)\,A(M)$.

If $\psi: M\to R^n$ is an isometric immersion into the Euclidean space with mean curvature vector H, then

$$\int_M |H|^2\, dM\geqq A_c(M). \tag{1.11}$$

Let $\psi: M\to S^n$ be an isometric immersion and let H and $\bar{H}$ be the mean curvature vectors of M in R^{n+1} and S^n respectively. Then, we have $\Delta\psi=2H=-2\psi+2\bar{H}$. For a fixed $g\in D^{n+1}$, we define $f: M\to R$ by $f=\langle\psi,g\rangle+1$. A direct computation gives

$$\Delta\log f=f^{-2}\{-2\langle\psi,g\rangle^2-2\langle\psi,g\rangle+2f\langle\bar{H},g\rangle-|g^T|^2\},$$

where g^T is the tangential part of the vector g with respect to M. Then $|g^T|^2=|g|^2-|g^N|^2-\langle g,\psi\rangle^2$, g^N being the component of g in the normal space to M in S^n. Hence

$$\Delta\log f=-1+\frac{1-|g|^2}{f^2}+\frac{2f\langle\bar{H},g\rangle+|g^N|^2}{f^2}.$$

Integrating this equation and taking into account (1.4), we get

$$A(0)=A(g)+\int_M\frac{2(\langle\psi,g\rangle+1)\langle\bar{H},g\rangle+|g^N|^2}{(\langle\psi,g\rangle+1)^2}\,dM. \tag{1.12}$$

2. Minimal immersions by the first Eigenfunctions

Let M be a compact surface. If ds_k^2 is a metric on M, we will denote by Δ_k, dM_k, $\lambda_1(M_k)$ and $A(M_k)$ the Laplacian, the canonical measure, the first non-zero eigenvalue and the area corresponding to ds_k^2.

Theorem 1. *For each conformal structure on a compact surface, there exists at most one metric which admits an isometric immersion into some unit sphere by the first eigenfunctions.*

Proof. Suppose that there exist two conformal metrics on M, say ds_1^2 and ds_2^2, admitting minimal immersions by the first eigenfunctions into some unit spheres. Let $\psi_k\colon M \to S^{n_k}$, $k=1,2$, be such immersions that we can consider full. Put $ds_2^2 = 2F_2\, ds_1^2$, F_2 being a positive function.

By using (1.9) we have $g \in D^{n+1}$ such that the conformal immersion $\psi_3 = g \circ \psi_2\colon M \to S^{n_2}$ satisfies

$$\int_M \psi_3\, dM_1 = 0. \tag{2.1}$$

We denote by ds_3^2 the induced metric on M from the immersion ψ_3 and put $ds_3^2 = 2F_3\, ds_1^2$.

From (2.1) and using the minimum principle for λ_1, we have

$$\begin{aligned} \lambda_1(M_1)\, A(M_1) &= \lambda_1(M_1) \int_M |\psi_3|^2\, dM_1 \\ &\leqq \int_M |\nabla\psi_3|^2\, dM_1 = \int_M |\nabla\psi_3|^2\, dM_3, \end{aligned} \tag{2.2}$$

where the last equality holds from the conformal invariance of the Dirichlet integral.

Since the left side in (1.8) is invariant under conformal transformations of the sphere, we have that

$$\int_M |\nabla\psi_3|^2\, dM_3 = 2A(M_3) \leqq 2\int_M |H_3|^2\, dM_3 = 2\int_M |H_2|^2\, dM_2, \tag{2.3}$$

where H_k is the mean curvature vector in the Euclidean space associated to ψ_k. Now, one has the relations $\Delta_2\psi_2 = 2H_2 = -\lambda_1(M_2)\psi_2$ because the immersion ψ_2 is by the first eigenfunctions, and so

$$2\int_M |H_2|^2\, dM_2 = \lambda_1(M_2)\, A(M_2). \tag{2.4}$$

From (2.2), (2.3) and (2.4) one gets $\lambda_1(M_1)\, A(M_1) \leqq \lambda_1(M_2)\, A(M_2)$. Now, if we change the roles of the metrics ds_1^2 and ds_2^2, we obtain the equality. As a consequence, the inequalities (2.2) and (2.3) become equalities. It follows, from the equality in (2.2), that the linear functions of the immersion ψ_3 are eigenfunctions for the metric ds_1^2 associated to the eigenvalue $\lambda_1(M_1)$, that is

$$\Delta_1\psi_3 = -\lambda_1(M_1)\psi_3. \tag{2.5}$$

On the other hand, the equality in (2.3) implies that ψ_3 is a minimal immersion into S^{n_2}, that is

(2.6) $$\Delta_3 \psi_3 = -2\psi_3.$$

Now, our hypothesis about the metric ds_1^2 say that $\lambda_1(M_1)=2$. This fact and the relations $\Delta_3=(2F_3)^{-1}\Delta_1$, (2.5) and (2.6) imply that $2F_3=1$. Hence $ds_1^2=ds_3^2$ and $\psi_3: M\to S^{n_2}$ is an isometric minimal immersion by the first eigenfunctions with respect to the metric ds_1^2.

So, we have two minimal immersions, say ψ_2 and ψ_3, by the first eigenfunctions such that $\psi_3=g\circ\psi_2$ for a certain $g\in D^{n_2+1}$. As $A(M_1)=A(M_3)=A(M_2)$, from (1.12) we conclude that

$$0=\int_M \frac{|g^{N_2}|^2}{(\langle\psi_2, g\rangle+1)^2}\, dM_2,$$

where g^{N_2} is the component of g in the normal space to M in S^{n_2} corresponding to the immersion ψ_2. Thus $g^{N_2}=0$ and a direct computation shows that the Hessian with respect to ds_2^2 of the function $f: M\to R$ defined by $f=\langle\psi_2, g\rangle$ is the bilinear form $-f\,ds_2^2$. Now, making use of the Obata theorem, we have the following alternative:

i) $f=0$ and, so, $g=0$ because the immersion ψ_2 was supposed full. Then $\psi_2=\psi_3$ and $ds_1^2=ds_2^2$.

ii) $f\neq 0$ and, so, (M, ds_2^2) the unit 2-sphere endowed with the standard metric and $\psi_2: M\to S^2$ is the identity map. Since the metric ds_1^2 is induced from the diffeomorphism $\psi_3=g\circ\psi_2: M\to S^2$, (M, ds_1^2) is also the standard 2-sphere. However, in this case, it can occur that $2F_2\neq 1$.

Thus, the proof of the theorem is completed.

Remark 1. In Theorem 1, we only claim the unicity of the metric for each conformal structure, but not the unicity of the minimal immersions by the first eigenfunctions. In fact, there could exist several of such immersions even with different full codimensions. The first consequence of Theorem 1 is that it permit us to hope a reasonable answer to the following problem:

"Classify all compact Riemannian surfaces admitting a minimal immersion into a sphere by the first eigenfunctions".

Ogiue, [7], and Yau, [9], proposed to study the compact minimal hypersurfaces embedded into a sphere by the first eigenfunctions. More concretely, Yau conjectured that every minimal embedding is by the first eigenfunctions. So, if the Yau conjecture is true, it follows from Theorem 1 that two compact minimal surfaces embedded into S^3 with the same conformal type are isometric.

As S^2 and RP^2 have a unique conformal structure, we have that the only metric on these surfaces admitting a minimal immersion by the first eigenfunctions is the standard one. The square and equilateral flat tori are other known examples of surfaces having this kind of immersions.

Theorem 1 enables us to characterize the equalities in some inequalities obtained by Li and Yau in [6]. We will only state the results corresponding to the real projective plane.

Theorem 2. i) *For any metric ds^2 on RP^2*

$$\lambda_1(RP^2)\,A(RP^2) \leqq 12\pi.$$

Equality holds if and only if ds^2 is the standard metrie on RP^2.

ii) *Let M be a compact surface in R^n homeomorphic to RP^2 with mean curvature vector H, then*

$$\int_M |H|^2\, dM \geqq 6\pi.$$

Equality holds if and only if M is the image of the Veronese suface under a conformal transformation of the Euclidean space.

iii) *Let M be a compact surface homeomorphic to RP^2. If M is a minimal surface in some sphere S^n, then*

$$A(M) \geqq 6\pi$$

and the equality holds if and only if M is the Veronese surface.

Remark 2. Since the minimal deformations of a minimal compact surface in S^n have constant area, it follows from iii) that, if $\psi_t\colon RP^2 \to S^n$ is a deformation of the Veronese surface by minimal surfaces, then ψ_t is the Veronese embedding for all t. However, if we consider the two-sheeted covering of the Veronese surface which is a minimal immersion of the standard 2-sphere by the second eigenfunctions, there exists minimal deformations $\psi_t\colon S^2 \to S^4$ of this immersion such that the induced metric on S^2 from ψ_t is not the standard one for $t \neq 0$ (see the Tjaden example in [4]).

Another result of Li and Yau which can be improved by making use of Theorem 1 is the following

Corollary 3. *Let ds_1^2 and ds_2^2 two conformal metrics on a compact surface M. Put $ds_2^2 = 2F\,ds_1^2$. Suppose that the metric ds_1^2 admits a minimal immersion in some unit sphere by the first eigenfunctions and let $\psi_2\colon (M, ds_2^2) \to S^n$ a minimal isometric immersion. Then*

$$A(M_2) \geqq A(M_1).$$

If the equality holds (M, ds_1^2) and (M, ds_2^2) are isometric and, provided that (M, ds_1^2) is not the standard unit 2-sphere, $2F = 1$.

Now, we will study the problem of the existence of minimal immersions by the first eigenfunctions in the special case where M is an orientable compact surface of genus one. In fact, we will solve it when M is immersed as a hypersurface.

Theorem 4. *Let M be a minimal torus immersed into S^3 by the first eigenfunctions. Then M is the Clifford torus.*

Proof. Suppose that $\psi_1\colon M \to S^3$ is a minimal immersion by the first eigenfunctions and let ds_1^2 be the induced metric on M from ψ_1. We denote by $\psi_2\colon M \to S^3$ the polar surface of M, that is, $\psi_2(p)$ is the unit normal vector of the immersion ψ_1 at the point $p \in M$. Then ψ_2 is a minimal immersion and, if ds_2^2 is the induced metric from ψ_2, we have $ds_2^2 = (1-K)\,ds_1^2$, K being the Gauss curvature for the metric ds_1^2 (see [5]). So, the corresponding canonical measures are related by $dM_2 = (1-K)\,dM_1$. Integrating and making use of the

Gauss-Bonnet theorem, we get

$$A(M_2)=A(M_1).$$

From Corollary 3 one has $1-K=1$. Hence ds_1^2 is a flat metric and, so, ψ_1 is the Clifford embedding.

Remark 3. Lawson conjectured, [9], that the only minimally embedded torus into S^3 is the Clifford torus. From Theorem 4 it follows that, if the Yau conjecture is true, then the same holds for the Lawson one.

In the general case, if M is a minimal fully immersed torus into S^n by the first eigenfunctions, from a result of Besson in [2], we have that $n\leqq 5$. Also, Yang and Yau in [8] have shown that $A(M)<8\pi$ and, hence, M is embedded into S^n, [6].

3. Conformal area of compact surfaces of genus one. Applications

Let (M,ds^2) be an orientable compact surface of genus one. There exists a metric conformal to ds^2 with Gauss curvature identically zero. So, (M,ds^2) is conformally equivalent to a flat torus $(R^2/\Gamma, ds_0^2)$ where Γ is a 2-dimensional lattice and ds_0^2 is the metric on R^2/Γ induced from the Euclidean metric on R^2. Moreover, it is well-known that each flat torus is isometric, up dilatations, to a flat torus $T(a,b)=R^2/\Gamma(a,b)$ where $\Gamma(a,b)$ is the lattice generated by $\{(1,0),(a,b)\}$ with $0\leqq a\leqq\frac{1}{2}$, $b>0$ and $a^2+b^2\geqq 1$ (see [1]).

Also, one can see in [1, p. 146] that the eigenvalues of the Laplacian of $(T(a,b), ds_0^2)$ are given by

$$\lambda_{pq}=4\pi^2\left\{q^2+\left(\frac{p-qa}{b}\right)^2\right\}, \tag{3.1}$$

where $(p,q)\in S=\{(r,s)\in Z\times Z/s\geqq 0$ or $s=0$ and $r\geqq 0\}$. The induced functions on $T(a,b)$ from

$$f_{pq}(x,y)=\cos 2\pi\left\langle\left(q,\frac{p-qa}{b}\right),(x,y)\right\rangle \tag{3.2}$$

$$g_{pq}(x,y)=\sin 2\pi\left\langle\left(q,\frac{p-qa}{b}\right),(x,y)\right\rangle,$$

are eigenfunctions associated to λ_{pq}.

Proposition 5. *Let $\psi\colon T(a,b)\to S^n$ be a conformal immersion of $(T(a,b), ds_0^2)$ into some unit sphere. If $\int_{T(a,b)}\psi\, dM_0=0$, where dM_0 is the canonical measure associated to the metric ds_0^2, then*

$$A(\psi)=\tfrac{1}{2}\int_{T(a,b)}|\nabla\psi|^2\, dM_0\geqq\frac{4\pi^2 b}{1+b^2+a^2-a}$$

and the equality holds if and only if ψ is given by a subspace of the sum of eigenspaces corresponding to $\lambda_{01}, \lambda_{10}$ and λ_{11}.

Proof. Let us denote by $\bar{f}_{pq}, \bar{g}_{pq}$ the normalized eigenfunctions obtained from (3.2), and we have

$$\psi = \sum_{pq} \{A_{pq} \bar{f}_{pq} + B_{pq} \bar{g}_{pq}\}$$

with $(p,q) \in S$ and $A_{pq}, B_{pq} \in R^{n+1}$. So, taking into account (3.2), we get

$$\frac{1}{2\pi} \psi_x = \sum_{pq} \{-q A_{pq} \bar{g}_{pq} + q B_{pq} \bar{f}_{pq}\}$$

$$\frac{1}{2\pi} \psi_y = \sum_{pq} \left\{ -\frac{p-qa}{b} A_{pq} \bar{g}_{pq} + \frac{p-qa}{b} B_{pq} \bar{f}_{pq} \right\}.$$

Putting $a_{pq} = |A_{pq}|^2 + |B_{pq}|^2$, one has

$$\int_{T(a,b)} |\psi_x|^2 \, dM_0 = 4\pi^2 \sum_{pq} q^2 a_{pq},$$

$$\int_{T(a,b)} |\psi_y|^2 \, dM_0 = 4\pi^2 \sum_{pq} \left(\frac{p-qa}{b}\right)^2 a_{pq},$$

$$\int_{T(a,b)} \langle \psi_x, \psi_y \rangle \, dM_0 = 4\pi^2 \sum_{pq} q \frac{p-qa}{b} a_{pq}.$$

Now, since ψ is a conformal immersion into S^n one has $|\psi|^2 = 1$, $|\psi_x|^2 = |\psi_y|^2$ and $\langle \psi_x, \psi_y \rangle = 0$. Using the above equations and (3.1) we can write

$$\sum_{pq} \{b^2 q^2 - (p-qa)^2\} a_{pq} = 0, \tag{3.3}$$

$$\sum_{pq} q(p-qa) a_{pq} = 0 \tag{3.4}$$

$$\sum_{pq} a_{pq} = \int_{T(a,b)} dM_0 = b, \tag{3.5}$$

$$\frac{1}{2\pi^2} A(\psi) = \frac{1}{4\pi^2} \int_{T(a,b)} |\nabla \psi|^2 \, dM_0 = \sum_{pq} \left\{ q^2 + \left(\frac{p-qa}{b}\right)^2 \right\} a_{pq}. \tag{3.6}$$

As we assume that $\int_{T(a,b)} \psi \, dM_0 = 0$, we have $a_{00} = 0$, that is, the index (p,q) in (3.3), (3.4), (3.5) and (3.6) runs over $S - \{(0,0)\}$. Now, from (3.3), we obtain for a_{10}:

$$a_{10} = \sum_{pq} \{b^2 q^2 - (p-qa)^2\} a_{pq} \qquad (p,q) \in S - \{(0,0),(1,0)\}.$$

Using this equality, we rewrite (3.4), (3.5) and (3.6) as follows

$$\sum_{pq} q(p-qa) a_{pq} = 0, \tag{3.7}$$

$$\sum_{pq} \{1 + b^2 q^2 - (p-qa)^2\} a_{pq} = b, \tag{3.8}$$

$$\frac{1}{2\pi^2} A(\psi) = 2 \sum_{pq} q^2 a_{pq} \tag{3.9}$$

with $(p,q) \in S - \{(0,0),(1,0)\}$.

We find in (3.8) the following value for a_{01}:

$$a_{01}=\frac{b}{1+b^2-a^2}-\sum_{pq}\frac{1+b^2q^2-(p-qa)^2}{1+b^2-a^2}a_{pq}$$

where $(p,q)\in S-\{(0,0),(1,0),(0,1)\}$. Substituting such value for a_{01} in (3.7) and (3.9) we get

$$\sum_{pq}\left\{\frac{q(p-qa)}{b}+\frac{a(1+b^2q^2-(p-qa)^2)}{b(1+b^2-a^2)}\right\}a_{pq}=\frac{a}{1+b^2-a^2}, \tag{3.10}$$

$$\frac{1}{2\pi^2}A(\psi)=\frac{2}{1+b^2-a^2}\{b+\sum_{pq}(q^2-a^2q^2-1+(p-qa)^2)a_{pq}\} \tag{3.11}$$

with $(p,q)\in S-\{(0,0),(1,0),(0,1)\}$.

Finally, from (3.10), we have

$$a_{11}=\frac{1}{1+b^2+a^2-a}\{ab-\sum_{pq}((a^2+b^2+1)pq+a(1-p^2-q^2))a_{pq}\}$$

where $(p,q)\in S^*=S-\{(0,0),(1,0),(0,1),(1,1)\}$. This together with (3.11) give us

$$\frac{1}{2\pi^2}A(\psi)=\frac{2}{1+b^2+a^2-a}\{b+\sum_{pq}(p^2+q^2-pq-1)a_{pq}\}$$

with $(p,q)\in S^*$. The proof of Proposition is now immediate observing that $1+b^2+a^2-a>0$ and $(p^2+q^2-pq-1)a_{pq}\geqq 0$ for all $(p,q)\in S^*$.

The following immersions $\psi_{ab}\colon (T(a,b),ds_0^2)\to S^5$ provide us examples of conformal immersions (in fact homothetic) which satisfy

$$A(\psi_{ab})=\frac{4\pi^2 b}{1+b^2+a^2-a}. \tag{3.12}$$

We define ψ_{ab} by

$$\psi_{ab}=\frac{1}{(1+b^2+a^2-a)^{\frac{1}{2}}}((b^2+a^2-a)^{\frac{1}{2}}f_{10},(b^2+a^2-a)^{\frac{1}{2}}g_{10},$$
$$(1-a)^{\frac{1}{2}}f_{01},(1-a)^{\frac{1}{2}}g_{01},a^{\frac{1}{2}}f_{11},a^{\frac{1}{2}}g_{11})$$

with f_{pq}, g_{pq} as in (3.2). These immersions are full into either S^5 or S^3 according to $T(a,b)$ is oblique $(a\neq 0)$ or rectangular $(a=0)$. It is easy to see that they have constant mean curvature and that the only minimal are ψ_{01} and $\psi_{\frac{1}{2}\frac{\sqrt{3}}{2}}$ which correspond to the immersions by the first eigenfunctions of the square and equilateral flat tori respectively.

In the case $a=0$ of the rectangular tori, one can prove that ψ_{0b} is, up motions of S^n, the only conformal immersion ψ of $(T(0,b),ds_0^2)$ into S^n such that ψ is given by a subspace of the sum of eigenspaces associated to λ_{01}, λ_{10} and λ_{11}.

On the other hand, since we have

$$\int_{T(a,b)}\psi_{ab}\,dM_0=0$$

for each (a, b) and the measures dM_0 and the induced from ψ_{ab} are proportional, the origin $g=0$ is a critical point for the function $A(g \circ \psi_{ab})$ (see (1.5)). By using the second variational formula (1.6) it is easily seen that $g=0$ is a local maximum for $A(g \circ \psi_{ab})$ when

$$(a-\tfrac{1}{2})^2+b^2<\tfrac{9}{4}. \tag{3.13}$$

As a consequence of Proposition 5 we have:

Corollary 6. *Let M be a compact surface conformally equivalent to the flat torus $T(a,b)$. Then*

$$A_c(M) \geqq \frac{4\pi^2 b}{1+b^2+a^2-a}.$$

Proof. It is sufficient to observe that, from (1.9) and Proposition 5, for any conformal immersion ψ of $(T(a,b), ds_0^2)$ into S^n there exists a conformal transformation g of S^n which satisfies

$$A(g \circ \psi) \geqq \frac{4\pi^2 b}{1+b^2+a^2-a}.$$

Remark 4. Since for all compact surface M one has $A_c(M) \geqq 4\pi$, [6], the above Corollary has only significance whenever $\pi b \geqq 1+b^2+a^2-a$.

The following result extends the region in the modular space of genus one where Li and Yau have shown to be true the Willmore conjecture:

Corollary 7. *Let M be a surface of genus one in R^n. If M is conformally equivalent to a flat torus $T(a,b)$ with $(a-\frac{1}{2})^2+(b-1)^2 \leqq 1/4$, then*

$$\int_M |H|^2 \, dM \geqq 2\pi^2.$$

Proof. It is sufficient to use (1.11) and Corollary 6.

Note that, using (3.12) and Proposition 5, for each conformal immersion ψ of $(T(a,b), ds_0^2)$ into S^n, we have

$$\sup_{g \in D^{n+1}} A(g \circ \psi) \geqq A(\psi_{ab}).$$

Moreover, we know that $g=0$ is a local maximum for the function $A(g \circ \psi_{ab})$ for certain values of (a,b) given in (3.13). Now, we will prove that, under some conditions, this local maximum is a global maximum. In fact, we have

Theorem 8. *Let M be a compact surface conformally equivalent to a flat rectangular torus $T_b = T(0,b)$ with $b \leqq \sqrt{5/3}$. Then*

$$A_c(M) = \frac{4\pi^2 b}{1+b^2}.$$

Proof. From the above considerations, it only remains to prove that the function $A(g \circ \psi_{0b})$ attains its maximum at the point $g=0$. For it, if dM_b is the induced measure on T_b from ψ_{0b}, by adding $\int_{T_b} |\bar{H}|^2 \, dM_b$ to both sides of (1.12) we get

$$A(\psi_{0b})+\int_{T_b}|\bar{H}|^2\,dM_b=A(g\circ\psi_{0b})+\int_{T_b}\frac{|f\bar{H}+g^N|^2}{f^2}\,dM_b, \tag{3.14}$$

where $f=\langle\psi_{0b},g\rangle+1$.

Let us denote by $d\mu$ the following measure on T_b:

$$d\mu=\frac{|f\bar{H}+g^N|^2}{\int_{T_b}|f\bar{H}+g^N|^2\,dM_b}\,dM_b$$

which is well-defined whenever either $g\neq 0$ or $b>1$. Since in the case $b=1$ Theorem 8 is true from [6, Proposition 1], henceforth we will suppose that $b>1$. As $\int_{T_b} d\mu=1$ and $f>0$, from the Jensen inequality for the convex function $\phi(t)=t^{-2}$, we have

$$\int_{T_b} f^{-2}\,d\mu\geqq\Big(\int_{T_b} f\,d\mu\Big)^{-2},$$

that is

$$\int_{T_b}\frac{|f\bar{H}+g^N|^2}{f^2}\,dM_b\geqq\frac{\Big(\int_{T_b}|f\bar{H}+g^N|^2\,dM_b\Big)^3}{\Big(\int_{T_b} f|f\bar{H}+g^N|^2\,dM_b\Big)^2}.$$

So, by using (3.14), the proof is achieved if we prove that

$$\frac{\Big(\int_{T_b}|f\bar{H}+g^N|^2\,dM_b\Big)^3}{\Big(\int_{T_b} f|f\bar{H}+g^N|^2\,dM_b\Big)^2}\geqq\int_{T_b}|\bar{H}|^2\,dM_b \tag{3.15}$$

for all $g\in D^4$. But the immersion ψ_{0b} is given by

$$\psi_{0b}(x,y)=\frac{1}{(1+b^2)^{\frac{1}{2}}}\left(b\cos\frac{2\pi}{b}y,\,b\sin\frac{2\pi}{b}y,\,\cos 2\pi x,\,\sin 2\pi x\right) \tag{3.16}$$

for $0\leqq x\leqq 1$, $0\leqq y\leqq b$ and so it is easy to see that a subgroup of the isometries group of S^3 which is isomorphic to $S^1\times S^1$ acts transitively on $\psi_{0b}(T_b)$. Taking into account the invariance of $A(g\circ\psi_{0b})$ under motions of S^3, we can ourselves restrict to the points $g\in D^4$ of the form $g=(\alpha,0,\beta,0)$, $\alpha^2+\beta^2<1$.

A direct computation from (3.16) transform the inequality (3.15) that we want to prove in

$$F_b(\alpha,\beta)\geqq F_b(0,0),\qquad \alpha^2+\beta^2<1 \tag{3.17}$$

where $F_b(\alpha,\beta)=(u_b(\alpha,\beta))^3/(v_b(\alpha,\beta))^2$ and

$$u_b(\alpha,\beta)=\frac{b^2+1}{8b}\left(\frac{\alpha^2}{b}+b\beta^2\right)+\frac{(b^2-1)^2}{4b} \tag{3.18}$$

$$v_b(\alpha,\beta)=\frac{(3-b^2)\alpha^2}{8b}+\frac{(3b^2-1)b\beta^2}{8}+\frac{(b^2-1)^2}{4b}.$$

We have that, at least for $1<b<\sqrt{3}$, F_b can be defined for arbitrary (α,β). Moreover, in this case, there exists A, B, C, $D>0$ such that

$$F_b(\alpha,\beta)\geqq\frac{(At^2+B)^3}{(Ct^2+D)^2},\qquad t^2=\alpha^2+\beta^2.$$

So, $\lim_{t^2\to\infty} F_b(\alpha,\beta)=\infty$ and thus F_b attains its infimum at a critical point. On the other hand, if (α,β) is critical for F_b, we have $\partial F_b/\partial\alpha=\partial F_b/\partial\beta=0$ and hence

$$\begin{aligned}(3.19)\qquad &\alpha\{3(1+b^2)\,v_b(\alpha,\beta)-2(3-b^2)\,u_b(\alpha,\beta)\}=0\\ &\beta\{3(1+b^2)\,v_b(\alpha,\beta)-2(3b^2-1)\,u_b(\alpha,\beta)\}=0.\end{aligned}$$

If $\alpha\neq 0$ and $\beta\neq 0$, then we have $u_b(\alpha,\beta)=v_b(\alpha,\beta)=0$. But this is impossible from (3.18). Now, if $\alpha=0$ or $\beta=0$, (3.19) implies $\alpha=\beta=0$ provided that $b\leqq\sqrt{5/3}$. Then $(0,0)$ is the only critical point for F_b and, so, F_b attains its infimum at this point. Thus the proof is finished.

Remark 5. It is stated in one of the steps in the proof of Theorem 2 of [6] that, for the area function $A\colon D^{n+1}\to R$ associated to an equivariant embedding $\psi\colon M^m\to S^n$ from a compact homogeneous manifold into a sphere, there is no saddle points, and, so, A has at most one local maximum (note that although in Theorem 2 of [6] ψ is assumed to be minimal, this hypothesis is not used in this concrete step). If this assertion would be true, the proof of our Theorem 8 could be simplified as follows: "We know, from (3.13), that $g=0$ is a local maximum for $A(g\circ\psi_{0b})$ when $b<\sqrt{2}$. Since ψ_{0b} is an equivariant homogeneous embedding, then $g=0$ is the unique local maximum. As $A(\psi_{0b})>4\pi=$ maximum value of $A(g\circ\psi_{0b})$ at the infinity, then the area function attains its maximum at $g=0$." However, the proof of this assertion in [6] is not correct: The reasoning fails when it is assured that one can find a non-negative direction for the second variational formula at a saddle point, which is normal to the level set of the area function containing the saddle point. In fact, the origin $g=0$ is a saddle point for the area function $A(g\circ\psi_{0b})$, if $b>\sqrt{2}$.

Remark 6. Theorem 8 shows that the estimate (1.11) is not sufficient to solve the Willmore conjecture. In fact, we have $4\pi^2 b/1+b^2\geqq 2\pi^2$ only for $b=1$.

It seems natural to think, after Theorem 8, that the function $A(g\circ\psi_{ab})$ attains its maximum at the origin provided that $g=0$ is a local maximum. This leads us to propose the following conjecture:

Conjecture. Let M a compact surface conformally equivalent to a flat torus $T(a,b)$ with $(a-\frac{1}{2})^2+b^2<9/4$. Then

$$A_c(M)=\frac{4\pi^2 b}{1+b^2+a^2-a}.$$

Finally, we state the following consequence of Theorem 8 which extends Theorem 4 to an arbitrary codimension making a suitable restriction on the conformal type.

Corollary 9. *Let M be a compact surface conformally equivalent to a flat rectangular torus T_b with $1<b\leqq\sqrt{5/3}$. There exists no minimal immersions by the first eigenfunctions from M into any unit sphere.*

Proof. Suppose that there exists a metric ds^2 on T_b conformal to the flat metric ds_0^2 and a full conformal immersion $\psi: (T_b, ds_0^2)\to S^n$ which is isometric, minimal and by the first eigenfunctions with respect to ds^2. From (1.9) and Proposition 5, there exists $g\in D^{n+1}$ such that

$$A(g\circ\psi)\geqq A(\psi_{0b}). \tag{3.20}$$

On the other hand

$$A(\psi)\geqq A(g\circ\psi) \tag{3.21}$$

because ψ is minimal. Moreover, since ψ is minimal by the first eigenfunctions, we have $A(\psi)=A_c(T_b)$, that is, $A(\psi)=A(\psi_{0b})$ as follows by using Theorem 8. So, the inequalities (3.20) and (3.21) become equalities. The first one and Proposition 5 say us that $\psi_{0b}=g\circ\psi$ (recall the unicity of ψ_{0b}) and the second one implies $g=0$. So, $\psi=\psi_{0b}$ up to a motion of S^n. But ψ_{0b} is minimal only when $b=1$. This is a contradiction and, so, the proof is finished.

References

1. Berger, M., Gauduchon, P., Mazet, E.: Le spectre d'une variété Riemannienne. Lect. Notes Math., vol. 194. Berlin-Heidelberg-New York: Springer 1971
2. Besson, G.: Sur la multiplicité de la première valeur propre des surfaces Riemanniennes. Ann. Inst. Fourier, Grenoble **30**, 109–128 (1980)
3. Cheng, S.Y.: Eigenfunctions and nodal sets. Comment. Math. Helv. **51**, 43–55 (1976)
4. Kozlowski, M., Simon, U.: Minimal immersions of 2-manifolds into spheres. Math. Z. **186**, 383–391 (1984)
5. Lawson, H.B.: Complete minimal surfaces in S^3. Ann. Math. **92**, 334–374 (1970)
6. Li, P., Yau, S.T.: A new conformal invariant and its applications to the Willmore conjecture and first eigenvalue of compact surfaces. Invent. Math. 69, 269–291 (1982)
7. Ogiue, K.: Open problems. In: Ochiai, T. (ed.). Surveys in Geometry. Geometry of the Laplace operator (in Japanese) 1980/1981
8. Yang, P., Yau, S.T.: Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds. Ann. Sc. Super. Pisa **7**, 55–63 (1980)
9. Yau, S.T.: Problem Section. Seminar on Differential Geometry. Ann. Math. Stud., vol. 102. Princeton: Princeton University Press 1982

Oblatum 13-III-1985

Note added in proof

The following recent information may be useful. N. Ejiri has shown that the only superminimal compact surface immersed into S^4 by the first eigenfunctions is the standard sphere ("Calabi lifting and surface geometry in S^4", preprint). He has obtained a strong result about the area of minimal RP^2 into spheres as well ("Equivariant minimal immersions of S^2 into $S^{2m}(1)$", preprint). R.L. Bryant told us (private communication) he has proved most of our results in 3, including a solution for our conjecture. A. El Soufi and I. Said informed us (private communication) they have got suitable generalizations for higher dimension of the results about conformal area.