CHARGED COSMOLOGICAL DUST SOLUTIONS OF THE COUPLED EINSTEIN AND MAXWELL EQUATIONS

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Dedicated to Louis Nirenberg on the occasion of his 85th birthday

Abstract. It is well known through the work of Majumdar, Papapetrou, Hartle, and Hawking that the coupled Einstein and Maxwell equations admit a static multiple blackhole solution representing a balanced equilibrium state of finitely many point charges. This is a result of the exact cancellation of gravitational attraction and electric repulsion under an explicit condition on the mass and charge ratio. The resulting system of particles, known as an *extremely charged* dust, gives rise to examples of spacetimes with naked singularities. In this paper, we consider the continuous limit of the Majumdar–Papapetrou–Hartle–Hawking solution modeling a space occupied by an extended distribution of extremely charged dust. We show that for a given smooth distribution of matter of finite ADM mass there is a continuous family of smooth solutions realizing asymptotically flat space metrics.

1. Introduction

The purpose of this paper is to establish the existence of an infinite family of smooth solutions to the statically coupled Einstein and Maxwell equations within the Majumdar [20] and Papapetrou [24] metric describing the gravitational and electromagnetic fields generated from an extremely charged distribution of cosmological dust.

Consider a system of massive particles carrying electric charges of the same sign. In Newtonian theory, the inter-particle gravitational attraction and Coulomb repulsion both follow inverse square-power laws so that a perfect cancellation of the forces is reached when the ratios of masses and charges of the particles satisfy a balancing condition. Under such a condition, a system of particles, as a distribution of charged dust, is referred to as *extremely charged*. Thus, due to the balanced forces, a static star of any geometric shape may be formed with an extremely charged dust. It is obviously important to know whether the same conclusion may be reached in the context of general relativity when the Einstein equations are coupled with the Maxwell equations. The earliest study of the Einstein–Maxwell equations was carried out by Reissner and Nordström [35] who obtained a static solution to the coupled field equations in empty space, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric massive body and generalizes the Schwarzschild blackhole metric. Since the Reissner–Nordström solution concerns only an isolated single particle, it gives no clue to the above extremely charged dust question. At roughly the same time as Reissner and Nordström, Weyl [36] classified all Einstein

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metrics induced by any static axially symmetric distribution of matter and charge. The work of Weyl work did not generate as much interest as those by Schwarzschild, Reissner, and Nordström, due to its geometric restriction of axial symmetry. A dramatic development came in 1947 when Majumdar [20] and Papapetrou [24] independently extended Weyl's work to include an arbitrarily distributed system of extremely charged particles. In particular, the Majumdar–Papapetrou formulation gives rise to the electrostatic solutions of the Einstein–Maxwell equations without any symmetry restriction, which marvelously generalizes the Newtonian theory to the relativistic situation of multiple charged blackholes [11]. It has also been argued that, although astrophysical bodies are electrically neutral within a reasonable approximation, the Majumdar–Papapetrou solutions may provide simple quasi-static analogies for complex dynamical processes preventing asymmetries in gravitational collapses or collision of blackholes [15]. More recently, it has been recognized that the Majumdar–Papapetrou solution allows a natural extension to all higher dimensions [18, 19] which is known to be important to issues concerning unification of gravity with other forces in nature [21, 22, 26, 27, 28].

Within the Majumdar–Papapetrou framework of an extremely charged discrete dust model, the coupled Einstein–Maxwell equations reduce to a Poisson equation in the vacuum space so that its solution represents a multiply distributed point charges with masses [11]. When a continuous distribution of extremely charged matter is considered, the field equations reduce to a semilinear elliptic equation.

However, in the literature for the continuous case, much attention has been given to dust models with spherical star or shell structures [3, 9, 10, 12, 13, 14, 16, 17, 19, 34] with the focus on explicit construction of solutions. There has not yet been any study of the existence problem of the governing elliptic equation when the matter distribution is arbitrarily prescribed nor of the transition process from the continuous model to the discrete model. The purpose of the present paper is to establish an existence theory for this important gravitational system.

In the next section, we follow [18] to lay out the governing elliptic equation for extremely charged dusts with a continuous matter distribution, within the framework of Majumdar–Papapetrou. In Section 3, we consider the condition to be imposed on the mass or charge density which ensures a finite ADM mass. In Section 4, we use the methods in Ni [23] to establish an existence theory for the equation. We show that, in fact, the Majumdar–Papapetrou equation allows a continuous monotone family of finite ADM mass solutions labeled by their asymptotic values. In Section 5, we derive some asymptotic estimates for the solutions obtained which lead to the expected asymptotic flatness of the gravitational metrics. In Section 6, we develop an existence theory using an energy method [5, 4] that requires much weaker conditions on the decay rate of the mass or charge density function. In Section 7, we present some nonexistence results. In particular, we show that when the asymptotic value of a conformal metric is larger than some $\beta_0 > 0$, there can be no solution (whereas there are solutions when the asymptotic value is less than β_0). In the special situation when the mass or charge density function is radially symmetric, the set of permissible positive asymptotic values is precisely of the form $(0, \beta_0]$. This result complements the conditions in our existence theorems presented in Sections 4 and 6. Finally in Section 8, we draw our conclusions.

2. The Majumdar–Papapetrou metric

Consider the $d = n + 1$ dimensional Minkowski spacetime of signature $(- + \cdots +)$. Following [18], the metric element is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -V\mathrm{d}t^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad \mu,\nu = 0,1,\cdots,n, \quad i,j = 1,\cdots,n. \tag{2.1}$$

Here and in the sequel, since we are interested in static solutions, all field variables depend on the space coordinates (x^i) only. The universal gravitational constant and the speed of light are both taken to be unity. Using ∇_μ to denote the covariant derivative, A_μ the electromagnetic gauge potential, and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ the electromagnetic field, we can write the electromagnetic energy-momentum tensor as

$$4\pi E_{\mu\nu} = F_\mu^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{2.2}$$

Moreover, let U_μ be the underlying relativistic matter fluid d-velocity and ρ_e and ρ_m the associated electric charge and mass densities, respectively. Then the current density is expressed as

$$J_\mu = \rho_e U_\mu, \tag{2.3}$$

and the material energy-momentum tensor is written as

$$T_{\mu\nu} = \rho_m U_\mu U_\nu + M_{\mu\nu}, \tag{2.4}$$

where $M_{\mu\nu}$ is the perfect-fluid, pressure-p, stress tensor given by

$$M_{\mu\nu} = p(U_\mu U_\nu + g_{\mu\nu}). \tag{2.5}$$

With the above notation, along with the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ represented in terms of the Ricci tensor $R_{\mu\nu}$ and Ricci scalar curvature R, the coupled Einstein and Maxwell equations are

$$G_{\mu\nu} = 8\pi(T_{\mu\nu} + E_{\mu\nu}), \tag{2.6}$$

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu. \tag{2.7}$$

As usual, the Weyl–Majumdar–Papapetrou ansatz in higher dimensions is generalized to be

$$A_\mu = \phi\delta_\mu^0, \quad U_\mu = -\sqrt{V}\delta_\mu^0. \tag{2.8}$$

Thus, the conservation law $\nabla_\nu(T^{\mu\nu} + E^{\mu\nu}) = 0$, which is the relativistic analogue to the Euler equation, reads [8, 18]

$$\partial_i p + \frac{1}{2V}(\rho_m + p)\partial_i V - \frac{1}{\sqrt{V}}\rho_e\partial_i\phi = 0, \quad i = 1,\cdots,n, \tag{2.9}$$

which takes, after substituting the Weyl-type relation $V = V(\phi)$, the compressed form

$$\partial_i p + \left((\rho_m + p)\frac{V'}{2V} - \frac{\rho_e}{\sqrt{V}}\right)\partial_i\phi = 0, \quad i = 1,\cdots,n. \tag{2.10}$$

On the other hand, the higher dimensional generalization of the Majumdar–Papapetrou condition obtained in [18] reads

$$V(\phi)=\left(a\pm\sqrt{\frac{2(n-2)}{n-1}}\phi\right)^2, \tag{2.11}$$

for some positive constant a and the extremely charged perfect fluid condition [18], which follows from the compatibility of the Einstein and Maxwell equations, is

$$\rho_e=4\frac{\sqrt{V}}{V'}\left(\frac{n-2}{n-1}\rho_m+\frac{n}{n-1}p\right). \tag{2.12}$$

As a consequence of (2.11) and (2.12), one arrives [18] at

$$\rho_e=\pm\sqrt{\frac{2(n-2)}{n-1}}\left(\rho_m+\frac{n}{n-2}p\right). \tag{2.13}$$

In view of (2.10), (2.11), and (2.13), it is seen that there holds $(n-2)V\partial_i p=p\partial_i V$, which, upon an integration, gives us the relation [18]

$$p=CV^{\frac{1}{n-2}}, \tag{2.14}$$

where C is a constant. In a perfect-fluid star, we have $p=0$ at the boundary surface of the star. Thus, we arrive at $C=0$. In other words, $p=0$ everywhere and the fluid becomes a dust. In particular, we are led to the following general dimensional extremely charged dust condition [18] extending that of the Majumdar–Papapetrou solution given as

$$\rho_e=\pm\sqrt{\frac{2(n-2)}{n-1}}\rho_m. \tag{2.15}$$

In the bottom situation with $n=3$, we have the classical condition for the extremely charged cosmological dust

$$|\rho_e|=\rho_m. \tag{2.16}$$

Choose $U>0$ and set

$$V=\frac{1}{U^2}. \tag{2.17}$$

Following [21] and [18], we rewrite the metric element (2.1) diagonally as

$$\mathrm{d}s^2=-U^{-2}\mathrm{d}t^2+U^{\frac{2}{n-2}}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{2.18}$$

Then, under the condition (2.15), the Einstein–Maxwell equations are reduced to the following single nonlinear elliptic equation [18]

$$\Delta U+8\pi\left(\frac{n-2}{n-1}\right)\rho_m(x)U^{\frac{n}{n-2}}=0,\quad x\in\mathbb{R}^n. \tag{2.19}$$

A solution of this equation will give rise to gravitational and electric forces generated from the extremely charged cosmological dust distribution. In the subsequent sections, we aim at constructing solutions of the equation.

3. An ADM mass calculation

Recall that the ADM mass [1, 29] of the space $(\mathbb{R}^n, \{h_{ij}\})$ housed within the Minkowski spacetime of the metric element (2.1) is given by the formula

$$m_{\mathrm{ADM}} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{\partial B_r} (\partial_i h_{ij} - \partial_j h_{ii})\nu^j \,\mathrm{d}S_r, \tag{3.1}$$

where $\mathrm{d}S_r$ is the area element of ∂B_r ($B_r = \{x \in \mathbb{R}^n \,|\, |x| < r\}$), ν denotes the outnormal vector to ∂B_r, ω_k is the surface area of the standard unit sphere S^k, and the metric $\{h_{ij}\}$ is asymptotically Euclidean satisfying

$$h_{ij} = c\delta_{ij} + \mathrm{O}(|x|^{-(n-2)}), \quad \partial_k g_{ij} = \mathrm{O}(|x|^{-(n-1)}), \quad \partial_\ell\partial_k g_{ij} = \mathrm{O}(|x|^{-n}), \tag{3.2}$$

for $|x| \to \infty$ with $c > 0$. Inserting (2.18) into (3.1), integrating, and using (2.19), we obtain

$$\begin{aligned} m_{\mathrm{ADM}} &= -\frac{1}{2\omega_{n-1}} \lim_{r\to\infty} \int_{\partial B_r} \left(\frac{\partial}{\partial\nu} U^{\frac{2}{n-2}}\right) \mathrm{d}S_r \\ &= -\frac{1}{2\omega_{n-1}} \int_{\mathbb{R}^n} (\Delta U^{\frac{2}{n-2}})\,\mathrm{d}x \\ &= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \left(\frac{8\pi\rho_m}{(n-1)} U^{\frac{4}{n-2}} + \frac{2(n-4)}{(n-2)^2} U^{\frac{2(3-n)}{n-2}} |\nabla U|^2\right) \mathrm{d}x. \end{aligned} \tag{3.3}$$

From (3.3), we clearly see that, when $n \geq 4$, $m_{\mathrm{ADM}} \geq 0$ and $m_{\mathrm{ADM}} = 0$ if and only if $\rho_m \equiv 0$ and the conformal metric is flat, U =constant. This statement may be viewed as a much simplified (Majumdar–Papapetrou) version of the positive mass theorem [25, 30, 31, 32, 33, 37]. Curiously, though, (3.3) is not as clear in the classical dimension $n = 3$. Thus, we need to take a different path to recognize m_{ADM}.

Since (3.2) is translated into

$$U = U_\infty + \mathrm{O}(|x|^{-(n-2)}), \quad \partial_i U = \mathrm{O}(|x|^{-(n-1)}), \quad \partial_i\partial_j U = \mathrm{O}(|x|^{-n}), \tag{3.4}$$

as $|x| \to \infty$ for some $U_\infty > 0$, we have, in view of (2.19) again, and the decay estimates stated in (3.4),

$$\begin{aligned} m_{\mathrm{ADM}} &= -\frac{1}{(n-2)\omega_{n-1}} \lim_{r\to\infty} \int_{\partial B_r} U^{\frac{4-n}{n-2}} \frac{\partial U}{\partial\nu}\,\mathrm{d}S_r \\ &= -\frac{1}{(n-2)\omega_{n-1}} \lim_{r\to\infty} \left(\int_{\partial B_r} \left[U^{\frac{4-n}{n-2}} - U_\infty^{\frac{4-n}{n-2}}\right] \frac{\partial U}{\partial\nu}\,\mathrm{d}S_r + \int_{\partial B_r} U_\infty^{\frac{4-n}{n-2}} \frac{\partial U}{\partial\nu}\,\mathrm{d}S_r\right) \\ &= \frac{8\pi}{(n-1)\omega_{n-1}} U_\infty^{\frac{4-n}{n-2}} \int_{\mathbb{R}^n} \rho_m(x) U^{\frac{n}{n-2}}(x)\,\mathrm{d}x, \end{aligned} \tag{3.5}$$

which is indeed positive-definite and vanishes only when $\rho_m \equiv 0$ and the metric becomes globally flat. Thus, we have arrived at a rather transparent (simplified) version of the positive mass theorem within the Majumdar–Papapetrou metric in all dimensions, $n \geq 3$.

In view of (3.4) and (3.5), we obtain the finite integral requirement

$$\int_{\mathbb{R}^n} \rho_m(x)\,\mathrm{d}x < \infty. \tag{3.6}$$

In the subsequent sections, we will construct solutions of the Majumdar–Papapetrou equation (2.19), observing the condition (3.6).

4. Existence of solutions by a sub-supersolution method

Following the above discussion, we rewrite the Majumdar–Papapetrou equation over $\mathbb{R}^n$ $(n\geq 3)$ in the compressed form

$$\Delta u+K(x)u^{\frac{n}{n-2}}=0,\quad x\in\mathbb{R}^n, \tag{4.1}$$

where $K(x)$ is a smooth nonnegative function in $\mathbb{R}^n$ which is not identically zero and lies in $L^1(\mathbb{R}^n)$. That is,

$$\int_{\mathbb{R}^n}K(x)\,\mathrm{d}x<\infty. \tag{4.2}$$

So it is natural to assume that

$$K(x)=\mathrm{O}(|x|^{-\ell})\quad\text{as } |x|\to\infty \tag{4.3}$$

where ℓ is large enough. For existence, a convenient condition is

$$\ell>2. \tag{4.4}$$

From (4.3) and (4.4), we may assume that there is a function $K_0(r)>0$ $(r=|x|)$ such that

$$K(x)\leq K_0(r),\quad r>0;\quad K_0(r)=\mathrm{O}(r^{-\ell})\quad\text{as } r\to\infty;\quad \ell>2. \tag{4.5}$$

We may adapt the method of sub- and supersolutions developed in [23]. For this purpose, consider

$$u_{rr}+\frac{(n-1)}{r}u_r = -K_0(r)u^{\frac{n}{n-2}},\quad r>0, \tag{4.6}$$

$$u(0) = \alpha>0,\quad u_r(0)=0. \tag{4.7}$$

It is well known that (4.6)–(4.7) can be solved locally over $[0,R)$ where $R>0$ is finite or infinite. Integrating (4.6)–(4.7), we get

$$r^{n-1}u_r(r)=-\int_0^r\rho^{n-1}K_0(\rho)u^{\frac{n}{n-2}}(\rho)\,\mathrm{d}\rho,\quad 0<r<R. \tag{4.8}$$

Hence $u(r)$ decreases in $[0,R)$ if $u(r)$ remains positive-valued.

Following [23], we consider a test function v defined by

$$v(r)=\frac{1}{(1+r^2)^a},\quad a>0. \tag{4.9}$$

Then $v(0)=1$, $v_r(0)=0$, and

$$v_{rr}+\frac{(n-1)}{r}v_r=-\left(n-\frac{2(a+1)r^2}{(1+r^2)}\right)\frac{2a}{(1+r^2)^{a+1}},\quad r>0. \tag{4.10}$$

Thus, when

$$n-2(a+1)\geq 0\quad\text{or}\quad a\leq\frac{n-2}{2}, \tag{4.11}$$

v is superharmonic, $\Delta v<0$. We shall maintain the condition (4.11) in the sequel.

Let u be the unique local solution of (4.6)–(4.7) which remains positive over its maximal interval $[0, R)$. That is, the solution exists and stays positive in $[0, R)$ for some $R > 0$ so that $u(R) = 0$ if R is finite.

Define $\varphi(r) = u(r) - \alpha v(r)$, $r \in [0, R)$. Then $\varphi(0) = 0, \varphi_r(0) = 0$, and φ satisfies

$$\begin{aligned} \varphi_{rr} + \frac{(n-1)}{r}\varphi_r &= -K_0(r)u^{\frac{n}{n-2}} + \left(n - \frac{2(a+1)r^2}{(1+r^2)}\right)\frac{2a\alpha}{(1+r^2)^{a+1}} \\ &\geq -K_0(r)\alpha^{\frac{n}{n-2}} + \left(n - \frac{2(a+1)r^2}{(1+r^2)}\right)\frac{2a\alpha}{(1+r^2)^{a+1}} \\ &= \alpha\left[\left(n - \frac{2(a+1)r^2}{(1+r^2)}\right)\frac{2a}{(1+r^2)^{a+1}} - K_0(r)\alpha^{\frac{2}{n-2}}\right] \\ &\equiv Q(r), \quad r < R. \end{aligned} \tag{4.12}$$

We want to make $Q(r) > 0$ for all $r > 0$. To do so, we set

$$2(a+1) \leq \ell \tag{4.13}$$

for $\ell > 2$ given in (4.5). Combining (4.11) and (4.13), we may assume

$$a \leq \frac{1}{2}\min\{n-2, \ell-2\}. \tag{4.14}$$

In (4.5), we may assume without loss of generality that there is a constant $C_\ell > 0$ such that

$$K_0(r) \leq \frac{C_\ell}{(1+r)^\ell}, \quad r \geq 0. \tag{4.15}$$

In view of (4.11)–(4.15), we can find some $\alpha_0 > 0$ such that

$$Q(r) > 0, \quad r > 0, \quad \alpha \in (0, \alpha_0). \tag{4.16}$$

From (4.12) and (4.16), we have as before,

$$r^{n-1}\varphi_r(r) > \int_0^r \rho^{n-1}Q(\rho)\,\mathrm{d}\rho > 0, \quad 0 < r < R. \tag{4.17}$$

In particular, $\varphi(r)$ increases for $0 < r < R$ and $\varphi(0) = 0$ implies

$$\lim_{r\to R}\varphi(r) > 0 \quad \text{or} \quad \lim_{r\to R}u(r) = \lim_{r\to R}\varphi(r) + \alpha\lim_{r\to R}v(r) > 0. \tag{4.18}$$

This result establishes $R = \infty$. In other words, there is some $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ the problem (4.6)–(4.7) has a unique solution, say $u_\alpha(r)$ over $0 < r < \infty$ so that $\varphi_\alpha(r) = u_\alpha(r) - \alpha v(r)$ increases and $u_\alpha(r)$ stays positive but decreases for all $r > 0$. Consequently, we have the positive finite limit

$$\lim_{r\to\infty}u_\alpha(r) = \lim_{r\to\infty}\varphi(r) \equiv \beta_\alpha > 0. \tag{4.19}$$

As a solution of $\Delta u + K_0 u^{\frac{n}{n-2}} = 0$ in $\mathbb{R}^n$, u_α is a supersolution of (4.1) because the property $u_\alpha > 0$ and (4.5) imply that

$$-\Delta u_\alpha = K_0(|x|)u_\alpha^{\frac{n}{n-2}} \geq K(x)u_\alpha^{\frac{n}{n-2}}, \quad x \in \mathbb{R}^n. \tag{4.20}$$

Set $v_\alpha(x) = \beta_\alpha$. We see that v_α is a subsolution of (4.1) since $K(x) \geq 0$. However, we have seen that $v_\alpha < u_\alpha$. As a consequence, we have

Lemma 4.1. *Equation (4.1) has a solution u satisfying*

$$\beta_\alpha = v_\alpha \leq u \leq u_\alpha \leq \alpha, \quad x \in \mathbb{R}^n \tag{4.21}$$

$$u(x) \to \beta_\alpha \quad \text{as } |x| \to \infty. \tag{4.22}$$

Moreover, u is the unique minimal positive solution of (4.1) in the sense that if w is any other solution satisfying $w(x) \to \beta_\alpha$ as $|x| \to \infty$, then $w \geq u$.

Proof. We briefly sketch the proof since it is standard. We first solve

$$\begin{aligned} -\Delta u_{j+1} &= K u_j^{\frac{n}{n-2}} \text{ in } B_R(0) \\ u_{j+1} &= \beta_\alpha \text{ on } \partial B_R(0) \\ u_0 &= v_\alpha \end{aligned}$$

and observe that

$$v_\alpha \leq u_j \leq v_{j+1} < u_\alpha \text{ (respectively } w).$$

(Here we used that $w \geq \beta_\alpha$; see section 7 for a simple proof.) Hence $u_R = \lim_{j\to\infty} u_j$ solves

$$\begin{aligned} -\Delta u_R &= K u_R^{\frac{n}{n-2}} \text{ in } B_R(0) \\ u_R &= \beta_\alpha \text{ on } \partial B_R(0) \end{aligned}$$

Finally, in the limit as R tends to infinity, we obtain the unique minimal solution u, of (4.1) satisfying

$$v_\alpha \leq u \leq u_\alpha \text{ (respectively } w).$$

□

As a by-product of (4.19) and (4.21), we have the immediate result

$$u(x) \to \beta_\alpha \quad \text{as } |x| \to \infty. \tag{4.23}$$

Next observe that

$$u_\alpha^{(\beta)} = \left(\frac{\beta}{\beta_\alpha}\right) u_\alpha, \quad 0 < \beta \leq \beta_\alpha, \tag{4.24}$$

satisfies

$$u_\alpha^{(\beta)} \geq \beta, \quad u_\alpha^{(\beta)}(x) \to \beta \text{ as } |x| \to \infty, \tag{4.25}$$

and

$$-\Delta u_\alpha^{(\beta)} = \left(\frac{\beta}{\beta_\alpha}\right)^{-\frac{2}{n-2}} K_0 (u_\alpha^{(\beta)})^{\frac{n}{n-2}} \geq K (u_\alpha^{(\beta)})^{\frac{n}{n-2}}. \tag{4.26}$$

Combining (4.24)–(4.26), we see that we have obtained a sub- and supersolution pair, consisting of $v_\alpha^{(\beta)} \equiv \beta$ and $u_\alpha^{(\beta)}$, satisfying $v_\alpha^{(\beta)} \leq u_\alpha^{(\beta)}$ for each $0 < \beta \leq \beta_\alpha$. In particular, using the above procedure, we see that there is a minimal positive solution $u^{(\beta)}(x)$ for any $0 < \beta \leq \beta_\alpha$ which depends continuously on β.

In summary, we have

Theorem 4.2. *Suppose that the function $K(x) \geq 0$ satisfies*

$$K(x) = O(|x|^{-\ell}), \quad \text{as } |x| \to \infty, \quad \ell > 2. \tag{4.27}$$

Then there is a positive number $\beta_0 > 0$ such that the Majumdar–Papapetrou equation (4.1) has a continuum of minimal positive solutions $\{u^{(\beta)} \,|\, 0 < \beta < \beta_0\}$ satisfying

$$\lim_{|x|\to\infty} u^{(\beta)}(x) = \beta, \quad u^{(\beta)} \geq \beta. \tag{4.28}$$

Moreover $u^{(\beta)}$ depends continuously in β and is monotone increasing in β.

In Section 7 we will show that $\beta_0 < \infty$.

We next consider the asymptotic decay estimates of the solutions obtained in Theorem 4.2 as $|x| \to \infty$.

5. Asymptotic estimates

We need the following somewhat standard result concerning a Newton-type potential integral.

Lemma 5.1. *Define the function $v(x)$ by the Newton-type integral*

$$v(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^\sigma}\, dy, \tag{5.1}$$

where $f(x) = \mathrm{O}(|x|^{-\gamma})$ as $|x| \to \infty$ for some constants γ and $\sigma < n$ and

$$\int_{\mathbb{R}^n} |f(x)|\, dx < \infty. \tag{5.2}$$

Then we have

$$v(x) = \mathrm{O}(|x|^{-\delta}) \quad \text{as } |x| \to \infty, \tag{5.3}$$

where $\delta = \min\{\sigma, \gamma+\sigma-n\}$, provided that the exponents γ and σ satisfy the condition

$$\gamma + \sigma > n. \tag{5.4}$$

Proof. For fixed $x \in \mathbb{R}^n$, $x \neq 0$, we decompose $\mathbb{R}^n$ into $\mathbb{R}^n = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where

$$\begin{aligned}
\Omega_1 &= \left\{ y \in \mathbb{R}^n \,\middle|\, |y-x| \leq \frac{|x|}{2} \right\}, \\
\Omega_2 &= \left\{ y \in \mathbb{R}^n \,\middle|\, \frac{|x|}{2} \leq |y-x| \leq 2|x| \right\}, \\
\Omega_3 &= \left\{ y \in \mathbb{R}^n \,\middle|\, |y-x| \geq 2|x| \right\},
\end{aligned}$$

which results in $v(x) = v_1(x) + v_2(x) + v_3(x)$ with

$$v_i(x) = \int_{\Omega_i} \frac{f(y)}{|y-x|^\sigma}\, \mathrm{d}y, \quad i = 1,2,3. \tag{5.5}$$

On the other hand, using the condition on $f(x)$, we have the bound

$$|f(y)| \leq \frac{C}{1+|y|^\gamma} \leq \frac{C}{1+(|x|/2)^\gamma} \leq \frac{2^\gamma C}{1+|x|^\gamma}, \quad y \in \Omega_1, \tag{5.6}$$

in view of $|x| - |y| \leq |x|/2$ for $y \in \Omega_1$. Therefore,

$$
\begin{aligned}
|v_1(x)| &\leq \frac{2^\gamma C}{(1+|x|^\gamma)} \int_{|y-x|\leq \frac{|x|}{2}} \frac{\mathrm{d}y}{|y-x|^\sigma} \\
&= \mathrm{O}(|x|^{-(\gamma+\sigma-n)}) \quad \text{as } |x| \to \infty. \qquad (5.7)
\end{aligned}
$$

For $y \in \Omega_2$, we have

$$
\begin{aligned}
|v_2(x)| &\leq \frac{2^\sigma}{|x|^\sigma} \int_{\frac{1}{2}|x|\leq|y-x|\leq 2|x|} |f(y)|\,\mathrm{d}y \\
&\leq \frac{2^\sigma}{|x|^\sigma} \int_{\mathbb{R}^n} |f(y)|\,\mathrm{d}y \\
&= \mathrm{O}(|x|^{-\sigma}) \quad \text{as } |x| \to \infty. \qquad (5.8)
\end{aligned}
$$

Furthermore, using $||x-y| - |x|| \leq |y|$, we have

$$
\begin{aligned}
|v_3(x)| &\leq C \int_{|y-x|\geq 2|x|} \frac{\mathrm{d}y}{|y-x|^\sigma(1+|y|^\gamma)} \\
&\leq C \int_{|y-x|\geq 2|x|} \frac{\mathrm{d}y}{|y-x|^\sigma(1+[|y-x|-|x|]^\gamma)} \\
&\leq C \int_{|y-x|\geq 2|x|} \frac{\mathrm{d}y}{|y-x|^\sigma(1+|y-x|^\gamma/2^\gamma)} \\
&= \mathrm{O}(|x|^{-(\gamma+\sigma-n)}) \quad \text{as } |x| \to \infty, \qquad (5.9)
\end{aligned}
$$

using $||y-x| - |x|| \geq |y-x|/2$, which is the same as stated in (5.7) provided that $\gamma + \sigma > n$ holds.

Hence, summarizing (5.7)–(5.9), we see that (5.3) is established. □

We now turn our attention to (4.1). Let u be a positive solution of (4.1) produced in Theorem 4.2. Then there is a constant $\beta > 0$ such that

$$\lim_{|x|\to\infty} u(x) = \beta. \qquad (5.10)$$

Let $w(x)$ be given by the Newton potential associated with (4.1), i.e.,

$$w(x) = \frac{1}{n(n-2)\Omega_n} \int_{\mathbb{R}^n} \frac{K(y)u^{\frac{n}{n-2}}(y)}{|x-y|^{n-2}}\,\mathrm{d}y, \qquad (5.11)$$

where $\Omega_n = 2\pi^{n/2}/n\Gamma(n/2)$ is the volume of the unit ball in $\mathbb{R}^n$. Applying Lemma 5.1 with $\sigma = n-2$ to $w(x)$ and noting (5.10), we see that, with $\ell = \gamma$, the conditions (4.4) and (5.4) coincide. Therefore, we obtain

$$w(x) = \mathrm{O}(|x|^{-m}) \quad \text{as } |x| \to \infty, \quad m = \min\{n-2, \ell-2\}. \qquad (5.12)$$

On the other hand, since $h = u - w$ is a bounded harmonic function in view of (5.10) and (5.12), it must be a constant, which must be given by (5.10). In other words, we have established the relation

$$u(x) = \beta + \frac{1}{n(n-2)\Omega_n} \int_{\mathbb{R}^n} \frac{K(y)u^{\frac{n}{n-2}}(y)}{|x-y|^{n-2}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n. \qquad (5.13)$$

Differentiating (5.13), we have

$$
\begin{aligned}
|\partial_i u(x)| &\le \frac{1}{n\Omega_n}\int_{\mathbb{R}^n}\frac{K(y)u^{\frac{n}{n-2}}(y)}{|x-y|^{n-1}}\,\mathrm{d}y \\
&= \mathrm{O}(|x|^{-(m+1)}) \quad \text{as } |x|\to\infty, \quad i=1,\cdots,n, \qquad (5.14)
\end{aligned}
$$

in view of Lemma 5.1 again.

In order to get the decay estimate for D^2u near infinity, we may be tempted to differentiate (5.13) twice and apply Lemma 5.1 again. However, such an approach will not work because of the technical restriction $\sigma < n$ for the Newton-type integral (5.1) stated in Lemma 5.1. To overcome this difficulty, we set $|x| = R$, assume that R is sufficiently large, and rewrite (5.13) as

$$
u(x) = \beta + U(x) + V(x), \qquad (5.15)
$$

where

$$
U(x) = \int_{|y|\le 2R}\Gamma(x-y)f(y)\,\mathrm{d}y, \quad V(x) = \int_{|y|\ge 2R}\Gamma(x-y)f(y)\,\mathrm{d}y, \qquad (5.16)
$$

with $\Gamma(z) = 1/n(n-2)\Omega_n|z|^{2-n}$ and $f(y) = -K(y)u^{\frac{n}{n-2}}(y)$. Then $|\partial_i\partial_j\Gamma(z)| \le 1/\Omega_n|z|^n$.

First, we have by using $|x-y|\ge|y|-R$ and $f(y)=\mathrm{O}(|y|^{-\ell})$ for $|y|\to\infty$,

$$
\begin{aligned}
|\partial_i\partial_j V(x)| &\le \int_{|y|\ge 2R}|\partial_i\partial_j\Gamma(x-y)f(y)|\,\mathrm{d}y \\
&\le C\int_{|y|\ge 2R}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y \\
&\le 2^nC\int_{|y|\ge 2R}|y|^{-(\ell+n)}\,\mathrm{d}y \\
&= C_1R^{-\ell}, \quad i,j=1,\cdots,n. \qquad (5.17)
\end{aligned}
$$

In order to estimate the second derivatives of $U(x)$ in (5.15), we extend $f(y)$ such that $f(y)=0$ for $|y|>2R$ and use Lemma 4.2 in [7] to represent $\partial_i\partial_j U(x)$ as

$$
\begin{aligned}
\partial_i\partial_j U(x) &= \int_{|y|\le 3R}\partial_i\partial_j\Gamma(x-y)(f(y)-f(x))\,\mathrm{d}y - f(x)\int_{|y|=3R}\partial_i\Gamma(x-y)\nu_j(y)\,\mathrm{d}S_y \\
&\equiv U_1(x)+U_2(x), \quad i,j=1,\cdots,n, \qquad (5.18)
\end{aligned}
$$

where $(\nu_j(y))$ is the outnormal vector at y over the sphere $\{|y|=3R\}$ and $\mathrm{d}S_y$ is the area element.

In view of $|x-y|\ge||y|-|x||=2R$ for $|y|=3R$, we have

$$
\begin{aligned}
|U_2(x)| &\le |f(x)|\int_{|y|=3R}|\partial_i\Gamma(x-y)\nu_j(y)|\,\mathrm{d}S_y \\
&\le C|f(x)|R^{(n-1)}\int_{|y|=3R}\mathrm{d}S_y \\
&\le C_2R^{-\ell}. \qquad (5.19)
\end{aligned}
$$

To estimate $U_1(x)$, we let Ω_k ($k=1,2,3$) be defined earlier and write

$$\begin{aligned} U_1(x) &= \sum_{k=1}^{3}\int_{\Omega_k\cap\{|y|\leq 3R\}}\partial_i\partial_j\Gamma(x-y)(f(y)-f(x))\,\mathrm{d}y \\ &\equiv W_1(x)+W_2(x)+W_3(x). \end{aligned} \tag{5.20}$$

We impose the additional natural decay condition on K as follows,

$$\partial_i K(x)=\mathrm{O}(|x|^{-(\ell+1)})\quad\text{as } |x|\to\infty,\quad i=1,\cdots,n. \tag{5.21}$$

In view of (5.21), (5.14), and the relation $f=Ku^{\frac{n}{n-2}}$, we have

$$\partial_i f(x)=\mathrm{O}(|x|^{-(\ell+1)})\quad\text{as } |x|\to\infty,\quad i=1,\cdots,n. \tag{5.22}$$

Since for $y\in\Omega_1$, we have $|y|\leq|y-x|+|x|\leq 3R/2<2R$, we see that $W_1(x)$ satisfies the estimate

$$\begin{aligned} |W_1(x)| &\leq \int_{\{|x-y|\leq R/2\}\cap\{|y|\leq 2R\}}|\partial_i\partial_j\Gamma(x-y)(f(x)-f(y))|\,\mathrm{d}y \\ &\leq C\int_{|x-y|\leq R/2}\frac{|f(x)-f(y)|}{|x-y|^n}\,\mathrm{d}y \\ &\leq C\int_{|x-y|\leq R/2}\frac{|\nabla f(\xi)|}{|x-y|^{n-1}}\,\mathrm{d}y, \end{aligned} \tag{5.23}$$

where $\xi=x+t(y-x)$ for some $t\in[0,1]$. Hence $|\xi|\geq|x|-|y-x|\geq R-R/2=R/2$ and $|\nabla f(\xi)|\leq C_3R^{-(\ell+1)}$ for some constant $C_3>0$. Inserting this result into (5.23) and integrating, we obtain

$$|W_1(x)|\leq C_4R^{-\ell}. \tag{5.24}$$

To estimate the decay rate of $W_2(x)$, we first note that

$$\int_{\frac{R}{2}\leq|y-x|\leq 2R}|\partial_i\partial_j\Gamma(x-y)f(y)|\,\mathrm{d}y\leq\frac{2^n}{\Omega_nR^n}\int_{\mathbb{R}^n}|f(y)|\,\mathrm{d}y. \tag{5.25}$$

Moreover, in view of $f(x)=\mathrm{O}(R^{-\ell})$, we have

$$\begin{aligned} \int_{\frac{R}{2}\leq|y-x|\leq 2R}|\partial_i\partial_j\Gamma(x-y)f(x)|\,\mathrm{d}y &\leq \mathrm{O}(R^{-\ell})\frac{2^n}{\Omega_nR^n}\int_{\frac{R}{2}\leq|y-x|\leq 2R}\mathrm{d}y \\ &\leq 2^{2n}\mathrm{O}(R^{-\ell}). \end{aligned} \tag{5.26}$$

Combining (5.25) and (5.26), we obtain

$$|W_2(x)|\leq C_5R^{-\min\{\ell,n\}}. \tag{5.27}$$

For $W_3(x)$, we make a similar decomposition. First we have the estimate

$$\int_{|y-x|\geq 2R}|\partial_i\partial_j\Gamma(x-y)f(y)|\,\mathrm{d}y\leq\frac{1}{2^n\Omega_nR^n}\int_{\mathbb{R}^n}|f(y)|\,\mathrm{d}y. \tag{5.28}$$

Furthermore, by virtue of $f(x) = \mathrm{O}(R^{-\ell})$ again, we find

$$\int_{\{|y-x|\geq 2R\}\cap\{|y|\leq 3R\}} |\partial_i\partial_j\Gamma(x-y)f(x)|\,\mathrm{d}y \leq \frac{\mathrm{O}(R^{-\ell})}{2^n\Omega_n R^n}\int_{\{|y-x|\geq 2R\}\cap\{|y|\leq 3R\}} \mathrm{d}y$$

$$\leq \left(\frac{3}{2}\right)^n \mathrm{O}(R^{-\ell}). \tag{5.29}$$

Combining (5.28) and (5.29), we arrive at

$$|W_3(x)| \leq C_5 R^{-\min\{\ell,n\}}. \tag{5.30}$$

Consequently, we may now collect (5.19), (5.24), (5.27), and (5.30) and recall the decompositions (5.18) and (5.20) to obtain the decay estimate

$$|\partial_i\partial_j U(x)| \leq C_6|x|^{-\min\{\ell,n\}} \quad \text{as } |x|\to\infty, \quad i,j=1,\cdots,n. \tag{5.31}$$

Finally, inserting (5.31) and (5.17) into (5.15), we see that the second derivatives of the solution $u(x)$ satisfy the same decay estimates as $U(x)$.

Summarizing the above discussion, we can state

Theorem 5.2. *Let u be a solution obtained in Theorem 4.2 with $u(x) \to \beta > 0$ as $|x| \to \infty$ when the charge density function K satisfies the conditions (4.3) and (4.4). Then u enjoys the decay estimates*

$$u(x) - \beta = O(|x|^{-m}), \quad Du(x) = O(|x|^{-(m+1)}), \quad |x|\to\infty, \tag{5.32}$$

where $m = \min\{n-2, \ell-2\}$ and D denotes a generic derivative. Moreover, if the function K satisfies the additional condition (5.21), then D^2u obeys $(D^2u)(x) = O(|x|^{-(m+2)})$ as $|x| \to \infty$.

The asymptotic flatness condition (3.4) may be ensured with the assumption $\ell = n$ such that $m = n-2$.

6. Existence of solutions by an energy method

In this section, we shall develop an existence theory for the Majumdar–Papapetrou equation (4.1) using an energy method.

We assume that $K(x)$ satisfies (4.3) with $\ell > 1$. If the constant $\beta \geq 0$ defined below is positive, we assume in addition that (4.2) is fulfilled.

Choose a sequence $\{\theta_k\}$ of positive constants decreasing to zero and define the Hilbert space

$$\mathcal{H}^1_k = \{v \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} (|\nabla v|^2 + \theta_k v^2)\,\mathrm{d}x < \infty\},$$

with inner product

$$(v,w)_k = \int_{\mathbb{R}^n} (\nabla v\cdot\nabla w + \theta_k v w)\,\mathrm{d}x\ .$$

and norm $\|v\|_k = (v,v)_k^{\frac{1}{2}}$. For later use, define also the Hilbert space $\mathcal{H}_K$ as the completion of the smooth functions of compact support under the inner product

$$(v,w)_K = \int_{\mathbb{R}^n} (\nabla v\cdot\nabla w + Kvw)\ \mathrm{d}x\ .$$

We will be interested in the subset

$$\mathcal{A}_k = \{v \in \mathcal{H}^1_k : v \geq 0,\ \|v\|_k = 1\}\ .$$

Fix $v_0 \in C^\infty(\mathbb{R}^n) \cap \mathcal{H}^1_0 \cap L^\infty(\mathbb{R}^n)$, $||v_0||_0 \leq 1$, $v_0 \geq 0$. We define a sequence $v_k \in \mathcal{A}_k$, $k \geq 1$ inductively as follows: for a fixed constant $\beta \geq 0$, let $w_{k+1} \in \mathcal{H}^1_{k+1}$ be the unique solution of

$$-\Delta w + \theta_{k+1} w = K(v_k + \beta)^{\frac{n}{n-2}} \tag{6.1}$$

and set $v_{k+1} = \lambda_{k+1} w_{k+1}$ where $\lambda_{k+1} = \frac{1}{\|w_{k+1}\|_{k+1}}$. Then

$$v_{k+1} \in \mathcal{A}_{k+1} \text{ and } -\Delta v_{k+1} + \theta_{k+1} v_{k+1} = \lambda_{k+1} K(v_k + \beta)^{\frac{n}{n-2}}\ . \tag{6.2}$$

To see the legitimacy of the above procedure, we inductively assume that $v_k \in \mathcal{H}^1_k$ and $v_k \geq 0$ some $k \geq 1$.

Recall the Gagliardo–Nirenberg inequality

$$\left(\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \leq C \int_{\mathbb{R}^n} |\nabla v|^2\,\mathrm{d}x. \tag{6.3}$$

As a consequence of (4.2) and (6.3), we see that $K(v_k + \beta)^{\frac{n}{n-2}} \in L^2(\mathbb{R}^n)$ since

$$\begin{aligned}\int_{\mathbb{R}^n} K^2 (v_k + \beta)^{\frac{2n}{n-2}}\,\mathrm{d}x &\leq C_1 \int_{\mathbb{R}^n} K^2 (v^{\frac{2n}{n-2}} + \beta^{\frac{2n}{n-2}})\,\mathrm{d}x \\ &\leq C_2 \left(\int_{\mathbb{R}^n} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{n}{n-2}} + C_3 \int_{\mathbb{R}^n} K\,\mathrm{d}x < \infty.\end{aligned}$$

Using the Lax–Milgram theorem, we see that (6.1) has a unique solution, say w_{k+1}, in $\mathcal{H}^1_{k+1}$, as claimed.

We need to show that $w_{k+1} \geq 0$. To see this, we consider the weak form of (6.1), that is,

$$\int_{\mathbb{R}^n} (\nabla w_{k+1} \cdot \nabla \phi + \theta_{k+1} w_{k+1} \phi)\,\mathrm{d}x = \int_{\mathbb{R}^n} K(v_k + \beta)^{\frac{n}{n-2}} \phi\,\mathrm{d}x, \quad \phi \in \mathcal{H}^1_{k+1}. \tag{6.4}$$

Let $m > 0$ and observe that $\phi = (-m - w_{k+1})^+ \in \mathcal{H}^1_{k+1}$. Inserting this choice of ϕ into (6.4) gives

$$\int_{\{w_{k+1} < -m\}} \theta_{k+1} w_{k+1} \phi\,\mathrm{d}x \geq \int_{\mathbb{R}^n} K(v_k + \beta)^{\frac{n}{n-2}} \phi\,\mathrm{d}x \geq 0,$$

which is impossible unless the set $\{w_{k+1} < -m\}$ is of measure zero. Since m is arbitrary, we have shown that $w_{k+1} \geq 0$ a.e., which establishes the legitimacy of the construction of the sequence $\{v_k\}$ as stated.

We define

$$\Phi(v) = \int_{\mathbb{R}^n} K(v^+ + \beta)^{\frac{2n-2}{n-2}}\,\mathrm{d}x\ .$$

Note that for $v \in \mathcal{A}_k$, we have the uniform bound

$$\begin{aligned}\Phi(v) &\leq 2^{\frac{n}{n-2}} \int_{\mathbb{R}^n} K(v^{\frac{2n-2}{n-2}} + \beta^{\frac{2n-2}{n-2}})\,\mathrm{d}x \\ &\leq C\left(\beta^{\frac{2n-2}{n-2}} \int_{\mathbb{R}^n} K\,\mathrm{d}x + \|K\|_{L^n(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\nabla v|^2 \mathrm{d}x\right)^{\frac{n-1}{n}}\right) \\ &\leq C\left(\beta^{\frac{2n-2}{n-2}} \int_{\mathbb{R}^n} K\,\mathrm{d}x + \|K\|_{L^n(\mathbb{R}^n)}\right), \qquad (6.5)\end{aligned}$$

with a uniform constant $C > 0$.

Lemma 6.1. $\Phi(v_k) \leq \Phi(v_{k+1})$, $k = 0, 1, \ldots$

Proof. From (6.1), we have

$$\begin{aligned}&\int_{\mathbb{R}^n} (\nabla v_{k+1} \cdot \nabla(v_{k+1} - v_k) + \theta_{k+1} v_{k+1}(v_{k+1} - v_k))\,\mathrm{d}x \\ &= \lambda_{k+1}\left(\int_{\mathbb{R}^n} K(v_{k+1} + \beta)(v_k + \beta)^{\frac{n}{n-2}}\,\mathrm{d}x - \Phi(v_k)\right) \\ &\leq \lambda_{k+1}\left(\frac{n-2}{2n-2}\Phi(v_{k+1}) + \frac{n}{2n-2}\Phi(v_k) - \Phi(v_k)\right) \\ &= \frac{n-2}{2n-2}\lambda_{k+1}(\Phi(v_{k+1}) - \Phi(v_k)). \qquad (6.6)\end{aligned}$$

However, the left-hand side of (6.6) satisfies

$$\begin{aligned}&\int_{\mathbb{R}^n} (\nabla v_{k+1} \cdot \nabla(v_{k+1} - v_k) + \theta_{k+1} v_{k+1}(v_{k+1} - v_k))\,\mathrm{d}x \\ &\geq \frac{1}{2}(\|v_{k+1}\|_{k+1}^2 - \|v_k\|_k^2 + (\theta_k - \theta_{k+1})\|v_k\|_{L^2(\mathbb{R}^n)}^2) \geq 0. \qquad (6.7)\end{aligned}$$

□

Corollary 6.2. *Assume that β lies in the interval*

$$0 \leq \beta < \beta_0 \equiv \left(\frac{\int_{\mathbb{R}^n} K v_0^{\frac{2n-2}{n-2}}\,dx}{\int_{\mathbb{R}^n} K\,dx}\right)^{\frac{n-2}{2n-2}}. \qquad (6.8)$$

Then there is a positive constant C such that the sequence $\{\lambda_k\}$ lies in the interval

$$C\|K\|_{L^n(\mathbb{R}^n)}^{-\frac{n-2}{2n-2}} \left(\|K\|_{L^1(\mathbb{R}^n)} + \|K\|_{L^n(\mathbb{R}^n)}\right)^{-\frac{n}{2n-2}} \leq \lambda_{k+1} \leq \lambda_0 = \Phi(v_0)^{-1}\left(1 - \frac{\beta}{\beta_0}\right)^{-1}.$$

Proof. From (6.2) and Lemma 6.1, we have

$$
\begin{aligned}
1 &\geq \int_{\mathbb{R}^n} (\nabla v_{k+1} \cdot \nabla v_k + \theta_{k+1} v_{k+1} v_k)\,\mathrm{d}x \\
&= \lambda_{k+1} \left(\Phi(v_k) - \beta \int_{\mathbb{R}^n} K (v_k + \beta)^{\frac{n}{n-2}}\,\mathrm{d}x \right) \\
&\geq \lambda_{k+1} \left(\Phi(v_k) - \beta \left(\int_{\mathbb{R}^n} K\,\mathrm{d}x \right)^{\frac{n-2}{2n-2}} \Phi(v_k)^{\frac{n}{2n-2}} \right) \\
&\geq \lambda_{k+1} \Phi(v_0) \left(1 - \frac{\beta}{\beta_0} \right).
\end{aligned}
$$

Similarly, in view of $v_{k+1} \in \mathcal{A}_{k+1}$, the inequality (6.3), and the bound (6.5), we also have

$$
\begin{aligned}
1 &= \lambda_{k+1} \int_{\mathbb{R}^n} K v_{k+1} (v_k + \beta)^{\frac{n}{n-2}}\,\mathrm{d}x \\
&\leq \lambda_{k+1} \|K\|_{L^n(\mathbb{R}^n)}^{\frac{n-2}{2n-2}} \left(\int_{\mathbb{R}^n} v_{k+1}^{\frac{2n}{n-2}}\,\mathrm{d}x \right)^{\frac{n-2}{2n}} \Phi(v_k)^{\frac{n}{2n-2}} \\
&\leq \lambda_{k+1} C \|K\|_{L^n(\mathbb{R}^n)}^{\frac{n-2}{2n-2}} \left(\|K\|_{L^1(\mathbb{R}^n)} + \|K\|_{L^n(\mathbb{R}^n)} \right)^{\frac{n}{2n-2}},
\end{aligned}
$$

for a uniform constant $C > 0$. $\square$

In order to prevent the sequence $\{\lambda_k\}$ from diverging to infinity or trivializing at zero, from now on, we assume β lies in the interval (6.8) as stated in Corollary 6.2.

Corollary 6.3. $\frac{1}{2}||v_{k+1} - v_k||_{k+1}^2 \leq \frac{n-2}{2n-2} \lambda_{k+1} (\Phi(v_{k+1}) - \Phi(v_k)))$.

Proof. We have

$$
\begin{aligned}
\frac{1}{2}||v_{k+1} - v_k||_{k+1}^2 &= (v_{k+1}, v_{k+1} - v_k)_{k+1} + \frac{1}{2}(||v_k||_{k+1}^2 - ||v_{k+1}||_{k+1}^2) \\
&\leq (v_{k+1}, v_{k+1} - v_k)_{k+1} \\
&\leq \frac{n-2}{2n-2} \lambda_{k+1} (\Phi(v_{k+1}) - \Phi(v_k))),
\end{aligned}
$$

by (6.6) and Corollary 6.2 since $\|v_k\|_{k+1}^2 \leq \|v_k\|_k^2 = 1 = \|v_{k+1}\|_{k+1}^2$. $\square$

We next show the sequence $\{v_k\}$ is uniformly bounded and obtain some (non-optimal) decay estimates.

Proposition 6.4. *There is a uniform constant C such that for $k \geq 1$,*
(i) $\|v_k\|_{L^\infty(\mathbb{R}^n)} \leq C$,
(ii) $v_k(y) \leq C(1 + |y|)^{-a}$ *where* $a = \min\left(\frac{n-2}{2}, \frac{(n-2)}{n} l - \varepsilon\right)$, $(0 < \varepsilon < \frac{(n-2)}{n} l)$ *if* $\beta > 0$,
(iii) $v_k(y) \leq C(1 + |y|)^{-\frac{n-2}{2}}$ *if* $\beta = 0$.

Proof. Fix a ball $B_R = B_R(y)$ (of radius $R > 0$ and centered at $y \in \mathbb{R}^n$) in $\mathbb{R}^n$. Then, by standard local elliptic estimates, we have

$$\sup_{B_{\frac{R}{2}}} v_{k+1} \leq C\left(R^{-\frac{n}{p}}\|v_{k+1}\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|K(v_k+\beta)^{\frac{n}{n-2}}\|_{L^q(B_R)}\right), \quad p > 0, \quad q > \frac{n}{2}. \tag{6.9}$$

We first observe that $v_k(x) \to 0$ as $|x| \to \infty$ for all $k \geq 1$. In fact, since K is bounded and satisfies (4.2), we see that $K(v_0+\beta)^{\frac{n}{n-2}} \in L^p(\mathbb{R}^n)$ for any $p > 1$. For $p > n$, elliptic theory applied to (6.1) implies $v_1 \in W^{1,p}(\mathbb{R}^n)$. Thus $v_1(x) \to 0$ as $|x| \to \infty$. Iterating this argument, we see that $v_{k+1}(x) \to 0$ as $|x| \to \infty$ for $k = 0, 1, 2, \cdots$.

To prove (i), let $R = 1$, $p = \frac{2n}{n-2}$ and choose y such that $\sup v_{k+1} = v_{k+1}(y)$. Then applying (6.3) and $v_k \in \mathcal{A}_k$ ($k \geq 1$), we obtain

$$\sup v_{k+1} \leq C\left(1 + \varepsilon\left(\int_{B_1}(v_k)^{q+\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{1}{q}} + C(\varepsilon)\right), \tag{6.10}$$

for $\frac{n}{2} < q < n$ (for example $q = \frac{3n}{4}$). Therefore, we find from (6.10) and (6.3) the bound

$$\begin{aligned}\sup v_{k+1} &\leq C\left(1 + \varepsilon(\sup v_k)\left(\int_{B_1} v_k^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{1}{q}} + C(\varepsilon)\right)\\ &\leq C(1 + \varepsilon\sup v_k + C(\varepsilon)).\end{aligned} \tag{6.11}$$

Now fix $\varepsilon = \frac{1}{2C}$ to obtain

$$\sup v_{k+1} \leq C + \frac{1}{2}\sup v_k \tag{6.12}$$

for a uniform constant C. Iterating, we find

$$\sup v_{k+1} \leq 2C + 2^{-(k+1)}\sup v_0, \tag{6.13}$$

proving (i).

To prove (ii), let $2R = |y|$, $p = \frac{2n}{n-2}$. Then, from (6.9), part (i), and the boundedness of $\Phi(v_k)$, we have

$$\begin{aligned}v_{k+1}(y) &\leq C\left(|y|^{-(\frac{n-2}{2})} + |y|^{(2-\frac{n}{q}-\frac{q-1}{q}l)}\right)\\ &= C\left(|y|^{-(\frac{n-2}{2})} + |y|^{-(l-2+\frac{n-l}{q})}\right).\end{aligned} \tag{6.14}$$

If $l \geq n$, we let $q = \infty$ and the result follows immediately. If $n > \ell > \frac{n}{2}$, then $l - 2 + \frac{n-l}{q} > \frac{n-2}{2}$ for q sufficiently close to $\frac{n}{2}$ and the result follows. If $1 < l \leq \frac{n}{2}$, choose $\frac{1}{q} = \frac{2}{n} - \frac{\varepsilon}{n-l}$, completing the proof of (ii).

Finally if $\beta = 0$, we observe that the decay estimates of part ii. are still valid and if we have shown that $v_k(y) \leq C(1+|y|)^{-a}$, $a < \frac{n-2}{2}$, then by the argument of part (ii), we find

$$v_{k+1}(y) \leq C\left(|y|^{-(\frac{n-2}{2})} + |y|^{-(l-2+\frac{n-l}{q}+a\frac{n}{n-2})}\right),$$

so that $v_{k+1} \leq C(1+|y|)^{-b}$ with $b = \frac{n}{n-2}a + l(\frac{n-2}{n} - \varepsilon) > \frac{n}{n-2}a$. Hence in a finite number of steps we arrive at the decay rate $\frac{n-2}{2}$ for a (different) uniform constant C. □

Remark 6.5. The dependence of the constant C on v_0 comes from the sup norm estimate of part (i). In particular, from (6.13), we see that this dependence will disappear in the limit as k tends to infinity. Also if we only assume $\int_{\mathbb{R}^n} |\nabla v_0|^2 dx \leq 1$ (instead of $\|v_0\|_0 \leq 1$), then, from (6.6) and (6.7) with $k = 0$, we have

$$\begin{aligned} \frac{n-2}{2n-2}\lambda_1(\Phi(v_1) - \Phi(v_0)) &\geq \frac{1}{2}\left(1 - \int_{\mathbb{R}^n} |\nabla v_0|^2 \,\mathrm{d}x - \theta_1 \|v_0\|^2_{L^2(\mathbb{R}^n)}\right) \\ &\geq -\frac{1}{2}\theta_0 \|v_0\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{6.15}$$

Hence, in view of (6.15) and Lemma 6.1, we have

$$\Phi(v_{k+1}) \geq \Phi(v_1) \geq \Phi(v_0) - C\theta_0 \|v_0\|^2_{L^2(\mathbb{R}^n)}. \tag{6.16}$$

This will be used in the proof of Theorem 6.7 below.

We are now ready to state and prove our main existence theorem of this section.

Theorem 6.6. *The sequence $\{v_k\}$ has a subsequence converging locally uniformly in $C^{2+\alpha}$ to a smooth positive function v such that $u = v + \beta$ is a finite energy solution of*

$$-\Delta u = \lambda K u^{\frac{n}{n-2}} \quad in\ \mathbb{R}^n, \tag{6.17}$$

$$u(x) \to \beta \quad as\ |x| \to \infty, \tag{6.18}$$

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \leq 1, \tag{6.19}$$

where $\lambda > 0$. If $\beta = 0$, $u = O(|x|^{-(n-2)})$ while if $\beta > 0$, $u - \beta = O(|x|^{-a})$ where $a = \min(\frac{n-2}{2}, \frac{(n-2)}{n}l - \varepsilon)$, $(0 < \varepsilon < \frac{(n-2)}{n}l)$. Moreover, all estimates are independent of $\sup v_0$ by Remark 6.5.

Proof. By Lemma 6.1 and (6.5), $\{\Phi(v_k)\}$ converges to a finite limit, say M. Since the sequence $\{v_k\}$ is uniformly bounded in $\mathcal{H}_K \cap L^\infty(\mathbb{R}^n)$, we can choose subsequences $\{v_{k_i}\}$, $\{v_{k_i+1}\}$ which converge weakly in $\mathcal{H}_K \cap L^\infty(\mathbb{R}^n)$. By Corollary 6.3, both subsequences converge weakly to the same limit, say v. Using that each v_{k+1} satisfies (6.2) and λ_{k+1} stays uniformly bounded from above and below (Corollary 6.2), Proposition 6.4 and standard elliptic regularity estimates give that the convergence is locally uniformly in $C^{2+\alpha}(\mathbb{R}^n)$ and that $u = v + \beta \in C^\infty(\mathbb{R}^n)$ satisfies (6.17)–(6.19) and

$$\int_{\mathbb{R}^n} K(x) u^{\frac{2n-2}{n-2}}(x) \,\mathrm{d}x = M, \tag{6.20}$$

where $\lambda > 0$ is a limiting point of the sequence $\{\lambda_k\}$. The decay properties of $u = v+\beta$ follow from Proposition 6.4 if $\beta > 0$ and using a bootstrap argument as in part (iii) of Proposition 6.4 in case $\beta = 0$. □

We next use Theorem 6.6 to solve the global constrained variational problem associated with equations (6.17), (6.18). Consider the variational problem

$$M_\beta := \max\left\{\Phi(v)\;\Big|\; v\in\mathcal{H}_K,\ \int_{\mathbb{R}^n}|\nabla v|^2\,\mathrm{d}x\le 1\right\}. \tag{6.21}$$

As shown earlier, M_β is always well-defined and the question is whether M_β is achieved.

Theorem 6.7. *Let K be a smooth nonnegative function satisfying $K=O(|x|^{-\ell})$, $\ell>1$. If $\beta>0$, assume (4.2) in addition. For β in the interval (6.8), the variational problem M_β has a finite energy solution v satisfying (6.17),(6.18),(6.19) with the decay properties described in Theorem 6.6.*

Proof. Let $\overline{v}^{(m)}$ be a maximizing sequence for M_β, that is $\Phi(\overline{v}^{(m)})\ge M_\beta-\frac{1}{m}$, $m=1,2,\cdots$. Without loss of generality we may assume $\overline{v}^{(m)}\in C^\infty(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $\overline{v}^{(m)}\ge 0$, $\Phi(\overline{v}^{(m)})\ge\frac{M_0}{2}$. Given $m\ge1$, we choose in our iterative scheme $\theta_0=(m\|\overline{v}^{(m)}\|^2_{L^2(\mathbb{R}^n)})^{-1}$, $v_0^{(m)}=\overline{v}^{(m)}$ and construct the sequence $\{v_k^{(m)}\}$ converging to the solution $v^{(m)}$ (say) of Theorem 6.6. Then from Lemma 6.1 (see (6.16) and Remark 6.5),

$$\begin{aligned}\Phi(v^{(m)}) &\ge \Phi(v_1^{(m)})\ge\Phi(\overline{v}^{(m)})-C\theta_0\|\overline{v}^{(m)}\|^2_{L^2(\mathbb{R}^n)}\\ &\ge M_\beta-\frac{C+1}{m}.\end{aligned} \tag{6.22}$$

Since $\int_{\mathbb{R}^n}|\nabla v^{(m)}|^2\mathrm{d}x\le1$, the sequence $\{v^{(m)}\}$ is also a maximizing sequence of solutions satisfying

$$-\Delta v^{(m)}=\lambda^{(m)}K(v^{(m)}+\beta)^{\frac{n}{n-2}}\ .$$

By our previous arguments, a subsequence converges locally uniformly in $C^{2+\alpha}(\mathbb{R}^n)$ to a nonnegative solution v of $-\Delta v=\lambda K(v+\beta)^{\frac{n}{n-2}}$ for some $\lambda>0$ which is a maximizer of the variational problem (6.21). □

A solution w of the original Majumdar–Papapetrou equation (4.1) may then be constructed from a solution u of the nonlinear eigenvalue problem (6.17) by setting

$$w=\lambda^{\frac{n-2}{2}}u. \tag{6.23}$$

7. Nonexistence results

For the Majumdar–Papapetrou type boundary value problem

$$\Delta u+K(x)u^{\frac{n}{n-2}} = 0,\quad u\ge0,\quad x\in\mathbb{R}^n, \tag{7.1}$$

$$\lim_{|x|\to\infty}u(x) = \beta,\quad \beta>0, \tag{7.2}$$

where $K(x)\ge0$ is a smooth function which is not identically zero and satisfies

$$\int_{\mathbb{R}^n}K(x)\,\mathrm{d}x<\infty;\quad K(x)=\mathrm{O}(|x|^{-\ell}),\quad \ell>2 \text{ or } \ell>1, \tag{7.3}$$

we have shown that a solution exists when β is not too large. Here we note that there may be no solution when β is sufficiently large. Without loss of generality, we choose the origin so that

$$K(0) = \max_{\mathbb{R}^n} K(x) > 0 \tag{7.4}$$

and set

$$K_0(r) = \min\{K(x) \,|\, |x| = r, x \in \mathbb{R}^n\}, \quad r \geq 0. \tag{7.5}$$

Lemma 7.1. *If the boundary value problem consisting of (7.1)–(7.2) has a solution, then the modified problem,*

$$\Delta v + K_0(r) v^{\frac{n}{n-2}} = 0, \quad \lim_{|x|\to\infty} v(x) = \beta, \quad v \geq 0, \tag{7.6}$$

has a radially symmetric solution.

Proof. Let u be a solution of (7.1)–(7.2). We first show that $u \geq \beta$. In fact, since $u(x) \to \beta$ as $|x| \to \infty$, we see that $w = \max\{0, \gamma - u\} = (\gamma - u)_+$ is of compact support for any $0 < \gamma < \beta$. Multiplying (7.1) by w integrating by parts, we have

$$-\int_{u\leq\gamma} |\nabla u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} K(x) u^{\frac{n}{n-2}} w \,\mathrm{d}x \geq 0, \tag{7.7}$$

which proves that the set $\{u < \beta\}$ must be empty as claimed.

On the other hand, using $K_0(r) = K_0(|x|) \leq K(x), x \in \mathbb{R}^n$ $(|x| = r)$, we have

$$-\Delta u \geq K_0 u^{\frac{n}{n-2}}. \tag{7.8}$$

In other words, $v^+ = u$ is a supersolution of (7.6). Taking $v^- \equiv \beta$, we see that v^- is a subsolution of (7.6) and $v^- \leq v^+$. We can now construct a radially symmetric solution v , of (7.6) satisfying $v^- \leq v \leq v^+$. This solution is just the minimal positive solution. As before, we solve

$$\begin{aligned} -\Delta v_{j+1} &= K_0 v_j^{\frac{n}{n-2}} \text{ in } B_R(0) \\ v_{j+1} &= \beta \text{ on } \partial B_R(0) \\ v_0 &= v^- \end{aligned}$$

and observe that since the Laplace operator commutes with rotations, the sequence$\{v_j\}$ is radially symmetric by uniqueness. Moreover,

$$v^- \leq v_j \leq v_{j+1} < v^+ \ .$$

Hence $v_R(r) = \lim_{j\to\infty} v_j(r)$ solves

$$\begin{aligned} -\Delta v_R &= K_0 v_R^{\frac{n}{n-2}} \text{ in } B_R(0) \\ v_R &= \beta \text{ on } \partial B_R(0) \end{aligned}$$

Finally, in the limit as R tends to infinity, we obtain a radially symmetric solution, v, of (7.6) satisfying $v^- \leq v \leq v^+$. □

Proposition 7.2. *Let $v(r)$ be a radially symmetric solution of the boundary value problem (7.6) where $K_0(0) > 0$. Then v is apriori bounded by a constant M (independent of β).*

Proof. Integrating (7.6) over the interval $(0, r)$, we see that $v = v(r)$ decreases. Hence

$$v(0) = \max\{v(r) \,|\, r \geq 0\} \tag{7.9}$$

If the proposition is false, then we may find a sequence of radially symmetric solutions of (7.6), say $\{v_j\}$, satisfying

$$M_j \equiv v_j(0) \to \infty \quad j = 1, 2, \cdots . \tag{7.10}$$

Now for fixed $j = 1, 2, \cdots$, define the rescaled function

$$w_j(x) = \frac{1}{M_j} v_j \left(\frac{x}{M_j^{\frac{1}{n-2}}} \right) \equiv \frac{1}{M_j} v_j(\tilde{x}). \tag{7.11}$$

Then $0 \leq w_j \leq 1 = w_j(0)$ and

$$-\Delta w_j = M_j^{-\frac{n}{n-2}} \tilde{\Delta} v_j = M_j^{-\frac{n}{n-2}} K_0(\tilde{x}) v_j^{\frac{n}{n-2}}(\tilde{x}) = K_0(\tilde{x}) w_j^{\frac{n}{n-2}}. \tag{7.12}$$

where $\tilde{\Delta}$ denotes the Laplace operator with respect to the variable $\tilde{x}$.

Using elliptic theory and passing to a subsequence if necessary, we see that the sequence $\{w_j\}$ converges (locally uniformly in $C^{2+\alpha}$) to a C^2 function w on $\mathbb{R}^n$ with $w(0) = 1$, which satisfies

$$\Delta w + K_0(0) w^{\frac{n}{n-2}} = 0. \tag{7.13}$$

On the other hand, however, by Theorem 1.1 in [6], we have $w \equiv 0$, which gives us a contradiction. □

Corollary 7.3. *Suppose that the function K satisfies (7.4). If the constant $\beta > 0$ in (7.2) is sufficiently large, the boundary value problem consisting of (7.1) and (7.2) will have no solution.*

Proof. If (7.1)–(7.2) has a solution, then (7.6) also has a radially symmetric solution as described in Lemma 7.1. Hence by Proposition 7.2, $\beta < M$. □

Theorem 7.4. *There exists $\beta_0 > 0$ so that there is a (minimal) solution of the boundary value problem (7.1) and (7.2) for any $0 \leq \beta < \beta_0$ and no solution for $\beta > \beta_0$. If $K(x)$ is radially symmetric about the origin with $K(0) > 0$, there is also a solution for β_0.*

Proof. Define β_0 be the supremum of the asymptotic values $\beta > 0$ such that the boundary value problem (7.1) and (7.2) has a solution. As we have shown in the proof of Theorem 4.2, there exists a minimal positive solution $u^{(\beta)}(x)$ for $0 \leq \beta < \beta_0$. Now assume that $K(x) = K_0(|x|)$ is radially symmetric about the origin with $K_0(0) > 0$. Then by Proposition 7.2, the minimal positive solutions $u^{(\beta)}(r)$ with asymptotic value β are all uniformly bounded independent of β. Hence $u^{(\beta_0)}(r) = \lim_{\beta \to \beta_0} u^{(\beta)}(r)$ is a C^2 solution of (7.6) with asymptotic value β_0. □

Next we consider a "shell"-star type solution of the equation (7.1). To motivate our discussion, recall that (7.1) allows a many-particle solution of the form [11]

$$u(x) = c + \sum_{j=1}^{N} \frac{\mu_j}{|x - p_j|^{n-2}}, \tag{7.14}$$

where $c, \mu_1, \cdots, \mu_N$ are positive constants. Such a solution determines the metric induced from a system of N extremely charged particles of masses $\mu_1, \cdots, \mu_N$ located at the points $p_1, \cdots, p_N \in \mathbb{R}^n$ so that the mass density is expressed as a sum of the Dirac distributions concentrated at $p_1, \cdots, p_N$. In particular, $u(x) \to \infty$ as $x \to p_j$ $(j = 1, \cdots, N)$.

Similar, an idealized "shell-star" type solution [10] represents a situation in which

$$K(x) = K_0(x)\delta(F(x)), \tag{7.15}$$

where $S = \{x \in \mathbb{R}^n \,|\, F(x) = 0\}$ is the shell which can be realized as the boundary surface of a bounded domain Ω, i.e., $S = \partial\Omega$, and $K_0(x) \geq 0$. Suggested by the afore-discussed many-particle solutions, we may imagine that $\Omega = \Omega_1 \cup \ldots \cup \Omega_k$ is the finite union of simply connected compact domains and look for a solution of (7.1)–(7.2) in $\mathbb{R}^n \setminus \Omega$ satisfying the exterior boundary condition

$$u(x) \to \infty, \quad x \to S = \partial\Omega. \tag{7.16}$$

However, one easily sees that this is hopeless. In fact,

Lemma 7.5. *There is no function u that is superharmonic ($\Delta u \leq 0$) and bounded below in $\mathbb{R}^n \setminus \Omega$ and satisfies (7.16).*

Proof. Choose the origin to lie in the interior of Ω. By adding a large constant to u, we may assume $u \geq \gamma > 0$ in $\mathbb{R}^n \setminus \Omega$. Then by the (weak) maximum principle, $u \geq \frac{C}{|x|^{n-2}}$ in $\mathbb{R}^n \setminus \Omega$ for any positive $C > 0$. This is clearly impossible. $\square$

8. Conclusions

In this paper, we have studied the Majumdar–Papapetrou equation describing an arbitrary continuously distributed extremely charged cosmological dust in static equilibrium governed by the coupled Einstein and Maxwell equations in a general $(n+1)$-dimensional spacetime and established several existence theorems for the solutions of the equation assuming that the ADM mass is finite. Among the results are the following.

(i) Assume that the mass or charge density vanishes at infinity at a rate no slower than $O(|x|^{-\ell})$ for some positive number ℓ. When $\ell > 2$, a positive solutions approaching a positive asymptotic value at infinity can be constructed by a sub- and supersolution method. When the condition is relaxed to $\ell > 1$, a positive solution with the same properties can be obtained by an energy method. The obtained solution and the structure of the equation allow us to establish the existence of a continuous monotone family of solutions labeled by their positive asymptotic values at infinity which are necessary for achieving asymptotic flatness.

(ii) Specific asymptotic rates of a solution obtained above have been established under various decay assumptions on the mass or charge density. In particular, if $\ell = n$ and the first derivatives of the mass or charge density decays like $O(|x|^{-(n+1)})$, then the conformal metric of a solution obeys all the decay rate assumptions in the classical definition of asymptotic flatness at infinity.

(iii) The set of possible asymptotic values of the conformal metrics determined by the Majumdar–Papapetrou equation is a bounded interval. In other words, there is a positive number $\beta_0 > 0$, such that there is no solution u satisfying the asymptotic condition $u(x) \to \beta$ as $|x| \to \infty$ when $\beta > \beta_0$ and there is a solution when $\beta < \beta_0$.

(iv) There is no such "shell-star" solution of the type that the solution gives rise to an asymptotic flat space and approaches infinity near the shell which is the boundary of a bounded domain so that the mass density is concentrated at the shell given by a Dirac distribution.

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