Problem 1

1. Find the Fourier series for \( f(x) = \cos ax \) on \([-\pi, \pi]\).

2. Find the Fourier transform of 
   \[ f(x) = e^{-|x|}. \]

Problem 2 Solve the following equation.

\[
\begin{cases}
  u_t = u_{xx} + u, & 0 < x < \pi, \ t > 0, \\
  u(0, t) = u(\pi, t) = 0, & t \geq 0, \\
  u(x, 0) = \varphi(x), & 0 \leq x \leq \pi.
\end{cases}
\]

Problem 3 Find the solvability condition for the equation

\[
\begin{cases}
  \frac{d^2 u}{dx^2} + u = f(x), \\
  u'(0) = A, \quad u'(\pi) = B.
\end{cases}
\]

Here \( A \) and \( B \) are fixed constants.

Problem 4 Suppose \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \). Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a solution of

\[
\begin{cases}
  -\Delta u + u^3 = 0, & x \in \Omega, \\
  a(x)u(x) = \varphi(x), & x \in \partial \Omega,
\end{cases}
\]

where \( a(x) \geq a_0 > 0 \) is continuous. Show that

\[ \sup_{\Omega} |u(x)| \leq \frac{1}{a_0} \max_{\partial \Omega} |\varphi(x)|. \]

Problem 5 Show that there is at most one solution of the following Dirichlet problem

\[
\begin{cases}
  \Delta u = f(x), & x \in \Omega, \\
  u(x) = \varphi(x), & x \in \partial \Omega.
\end{cases}
\]

Problem 6 Let \( \Omega = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, \ y > 0 \} \). Solve the following Dirichlet problem

\[
\begin{cases}
  \Delta u = f(x), & x \in \Omega, \\
  u = 0, & x \in \partial \Omega.
\end{cases}
\]