Analysis II
Final exam

Problem 1 Find the Fourier series for the following functions on $[-\pi, \pi]$.

1. $f(x) = \sin ax$.
2. $f(x) = \pi^2 - x^2$.

Problem 2 Let $f$ be a $C^3$ periodic function of period $2\pi$. Show that the Fourier series of $f$ converges uniformly to $f$, furthermore,

$$S_N f(x) - f(x) = O\left(\frac{1}{N^2}\right).$$

Problem 3 Suppose $f$ and $g$ are both continuous periodic functions of period $2\pi$. Calculate $\|f * g\|_{L^2}$ in terms of the Fourier coefficients of $f$ and $g$, where $(f * g)(x) = \int^{\pi}_{-\pi} f(x - y) g(y) dy$.

Problem 4 Find a function $f(x)$ such that $f \in L^1[0, 1]$ and $f \notin L^p[0, 1]$ for any $p > 1$.

Problem 5 State and prove the Cauchy-Schwartz inequality.

Problem 6 Let $\{f_n\}$ be a sequence of measurable functions converging pointwise to $f$. Assume that there exists an integrable function $g$ and a constant $K > 0$ such that

$$|f_n(x)| \leq K|g(x)|$$

for all $n$ and $x$. Show that

$$\lim_{n \to \infty} \int |f_n - f| d\mu = 0.$$