Problem 1 Find the Fourier series for the following functions on $[-\pi, \pi]$.

1. $f(x) = \cos ax$,
   \textbf{Solution:} $\cos ax = \frac{1}{2}(e^{iax} - e^{-iax})$.

2. $f(x) = x^2 - 3$.

Problem 2 Prove the Projection Theorem.

Problem 3 Let $f$ be a $C^k$, $k \geq 2$ periodic function of period $2\pi$. Show that the Fourier series of $f$ converges uniformly to $f$, furthermore,

$$S_N f(x) - f(x) = O\left(\frac{1}{N^{k-1}}\right).$$

\textbf{Solution:}

$$|S_N f(x) - f(x)| = \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x - y) - f(x)) \frac{\sin(N + 1/2)y}{\sin 1/2y} dy\right|$$

$$= \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) \sin(N + 1/2)y dy\right|,$$

Here $g(y) = \frac{f(x - y) - f(x)}{\sin 1/2y}$.

Show $g(y)$ is in $C^{k-1}$. Then apply integration by part $k - 1$ times.

Problem 4 Suppose $\mu(X) < \infty$ and $f_n$ is a sequence of bounded measurable functions on $X$. Show that if $f_n \to f$ uniformly, then in $L^1$

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu.$$

Give a counterexample if $\mu(X) = \infty$.

\textbf{Solution:}

$$\int_X |f - f_k| d\mu \leq \sup_X |f - f_k| \mu(X) \to 0.$$

Counterexample: $f_k(x) = \frac{1}{k}$.

Problem 5 Prove Monotone convergence theorem as a corollary of Fatou’s lemma.

Problem 6 Suppose $(X, \mathcal{F}, \mu)$ is a measure space and $f$ is integrable. Show that for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$

$$\int_E |f| d\mu < \epsilon.$$
Solution: Prove by contradiction. Suppose there exists $\epsilon < 0$ such that for any $\delta > 0$

$$
\int_E |f|d\mu \geq \epsilon
$$

for any $E \in \mathcal{F}$ with $\mu(E) < \delta$.

Choose a sequence $E_k \in \mathcal{F}$ such that $\mu(E_k) \leq \frac{1}{k}$. Let $f_k = f\chi_{E_k}$. Hence $f \to 0$ almost everywhere. By Dominated convergence theorem, we have

$$
\lim_{k \to \infty} \int_{E_k} |f|d\mu = \lim_{k \to \infty} \int |f_k|d\mu = 0.
$$

This contradicts the assumption that $\int_{E_k} |f|d\mu \geq \epsilon$. 
