Analysis II
Final exam

Problem 1 Find the Fourier series for the following functions on \([-\pi, \pi]\).

1. \(f(x) = \cos ax\),
   Solution: \(\cos ax = \frac{1}{2}(e^{iax} - e^{-iax})\).

2. \(f(x) = x^2 - 3\).

Problem 2 Prove the Projection Theorem.

Problem 3 Let \(f\) be a \(C^k, k \geq 2\) periodic function of period \(2\pi\). Show that the Fourier series of \(f\) converges uniformly to \(f\), furthermore,

\[ S_N f(x) - f(x) = O\left(\frac{1}{N^{k-1}}\right). \]

Solution:

\[
|S_N f(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - y) \frac{\sin(N + 1/2)y}{\sin 1/2y} dy \right|
\]

\[
= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) \sin(N + 1/2)y dy \right|.
\]

Here \(g(y) = \frac{f(x-y) - f(x)}{\sin 1/2y}\).

Show \(g(y)\) is in \(C^{k-1}\). Then apply integration by part \(k - 1\) times.

Problem 4 Suppose \(\mu(X) < \infty\) and \(f_n\) is a sequence of bounded measurable functions on \(X\). Show that if \(f_n \to f\) uniformly, then in \(L^1\)

\[
\lim_{n \to \infty} \int f_n d\mu = \int f d\mu.
\]

Give a counterexample if \(\mu(X) = \infty\).

Solution:

\[
\int_X |f - f_k|d\mu \leq \sup_X |f - f_k|\mu(X) \to 0.
\]

Counterexample: \(f_k(x) = \frac{1}{k}\).

Problem 5 Prove Monotone convergence theorem as a corollary of Fatou’s lemma.

Problem 6 Suppose \((X, F, \mu)\) is a measure space and \(f\) is integrable. Show that for any \(\epsilon > 0\) there exists \(\delta > 0\) such that whenever \(\mu(E) < \delta\)

\[
\int_E |f|d\mu < \epsilon.
\]
**Solution:** Prove by contradiction. Suppose there exists $\epsilon < 0$ such that for any $\delta > 0$

$$\int_E |f| d\mu \geq \epsilon$$

for any $E \in \mathcal{F}$ with $\mu(E) < \delta$.

Choose a sequence $E_k \in \mathcal{F}$ such that $\mu(E_k) \leq \frac{1}{k}$ and $E_k \subset E_{k-1}$. Let $f_k = f\chi_{E_k}$. Hence $f \to 0$ almost everywhere. By Dominated convergence theorem, we have

$$\lim_{k \to \infty} \int_{E_k} |f| d\mu = \lim_{k \to \infty} \int |f_k| d\mu = 0.$$

This contradicts the assumption that $\int_{E_k} |f| d\mu \geq \epsilon$. 

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