THE $\alpha$-INVARIANT ON $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$

Jian Song

Department of Mathematics
Columbia University, New York, NY 10027

§1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian[6], Tian and Yau[5] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau[11] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with nonpositive first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold $M$ with positive Chern class, Tian[6] proved that $M$ admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $\mathbb{CP}^2#\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2#2\overline{\mathbb{CP}^2}$. It would be also interesting to find the estimate of the $\alpha$ invariant for $\mathbb{CP}^2#\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2#2\overline{\mathbb{CP}^2}$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman kernel on polarized Kähler metrics to approximate plurisubharmonic functions and compute the $\alpha$-invariant of $\mathbb{CP}^2#2\overline{\mathbb{CP}^2}$. This gives an improvement of Abdesselem’s result[1]. More precisely, we shall show that:

**Theorem 1** $\alpha_G(CP^2#2\overline{CP^2})=\frac{1}{3}$.

Let $(M, \omega)$ be a compact Kähler manifold, where $\omega=\sqrt{-1}g_{ij}dz_i\wedge d\bar{z}_j$. We will also prove Tian’s conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M, \omega) = \left\{ \phi \mid \omega_\phi = \omega + \partial\bar{\partial}\omega > 0, \sup_M \phi = 0 \right\}.$$ 

Let $F_\omega$ and $J_\omega$ be the functionals defined on $P(M, \omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V} \int_M \phi\omega^n - \log\left(\frac{1}{V} \int_M e^{h_\omega-\phi}\omega^n\right).$$
\[ J_\omega(M) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial \phi \wedge \bar{\partial} \phi \wedge \omega^i \wedge \omega^{n-i-1}. \]

Assume \((M, \omega_{KE})\) is a Kähler-Einstein manifold with positive first Chern class and \(\text{Ric}(\omega_{KE}) = \omega_{KE}\), then for any \(\phi \in P(M, \omega_{KE})\), Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_\omega(\phi)-\frac{1}{V} \int_M \phi \omega^n}.
\]

Tian[9] also conjectured that \(\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)J_\omega(\phi)-\frac{1}{V} \int_M \phi \omega^n}\) for \(\delta > 0\) sufficiently small, if \(\phi\) is perpendicular to \(\Lambda_1\), the space of eigenfunctions of \(\omega_{KE}\) with eigenvalue one.

We shall prove:

**Theorem 2** Let \((M, \omega)\) be a Kähler manifold with positive first Chern class. Assume that \(\alpha(M) > \frac{n}{n+1}\), so that \(M\) admits a Kähler-Einstein metric \(\omega_{KE}\), and there exist constants \(\delta = \delta(n, \alpha(M))\) and \(C = C(n, \lambda_2(\omega_{KE}) - 1, \alpha(M))\) such that for any \(\phi \in P(M, \omega_{KE})\) which satisfies \(\phi \perp \Lambda_1\), we have:

\[
F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C
\]

Here \(\lambda_2(\omega_{KE})\) is the least eigenvalue of \(\omega_{KE}\) which is bigger than 1.

**Acknowledgements.** The author deeply thanks his advisor, Professor D.H. Phong for his constant encouragement and help. He also thanks Professor G. Tian for his suggestion on this work. This paper is part of the author’s future Ph.D. thesis in Math Department of Columbia University.

§2. Holomorphic approximation of psh

In this section, we will employ the technique in [7, 12] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem[7, 12].
Theorem 2.1 Let $M$ be a compact complex manifold of dimension $n$ and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = \text{Ric}(h)$. For each $m \in N$, $h$ induces a Hermitian metric $h_m$ on $L^m$. Let $\{S_0^m, S_1^m, \ldots, S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!} \omega^n_g$ is the volume form of $g$. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} \left\|S_i^m(x)\right\|^2_{h_m} = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \ldots$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any $k$:

$$\left\|\sum_{i=0}^{d_m-1} \left\|S_i^m(x)\right\|^2_{h_m} - \sum_{j<R} a_j(x)m^{n-j}\right\|_{C^k} \leq C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on $R, k$ and the manifold $M$.

Let

$$\tilde{\omega}_g = \omega_g + \theta \partial \bar{\partial} \phi > 0$$

$$\tilde{h} = h e^{-\phi}$$

Let $\tilde{h}_m$ be the induced Hermitian metric of $\tilde{h}$ on $L^m$, $\{\tilde{S}_0^m, \tilde{S}_1^m, \ldots, \tilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\tilde{h}_m} = \int_M \tilde{h}_m(S_1(x), S_2(x))dV_g.$$ 

By Theorem 2.1, we have

$$\sum_{i=0}^{d_m-1} \left\|\tilde{S}_i^m(x)\right\|^2_{h_m} = \tilde{a}_0(x)m^n + \tilde{a}_1(x)m^{n-1} + \tilde{a}_2(x)m^{n-2} + \ldots$$

$$= \left(\sum_{i=0}^{d_m-1} \left\|\tilde{S}_i^m(x)\right\|^2_{h_m}\right) e^{-m\phi}.$$
Thus

\[ \phi = \frac{1}{m} \log \left( \sum_{i=0}^{d_{m}-1} ||\tilde{S}_m^i(x)||^2_{h_m} \right) - \frac{1}{m} \log \left( \tilde{a}_0(x)m^n + \tilde{a}_1(x)m^{n-1} + \tilde{a}_2(x)m^{n-2} + \ldots \right) \]

As \( m \to +\infty \), we obtain

\[ \frac{1}{m} \log \left( \tilde{a}_0(x)m^n + \tilde{a}_1(x)m^{n-1} + \tilde{a}_2(x)m^{n-2} + \ldots \right) = \frac{1}{m} \log m^n(1 + \tilde{a}_1(x)m^{-1} + \tilde{a}_2(x)m^{-2} + \ldots) = \frac{n}{m} \log m + \frac{1}{m} \log(1 + O\left(\frac{1}{m}\right)) \to 0 \]

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

**Corollary 2.1**

\[ \left\| \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_{m}-1} ||\tilde{S}_m^i(x)||^2_{h_m} \right) \right\|_{C^k} \to 0, \text{ as } m \to +\infty. \]

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of \( L^m \).

### §3. Proof of Theorem 1

Let \( M \) be the blow-up of \( CP^2 \) at two points and \( \pi \) its natural projection. Without loss of generality, we may assume the two points are \( p_1 = [0,1,0] \) and \( p_2 = [0,0,1] \). Then \( M \) is a subvariety of \( CP^2 \times CP^1 \times CP^1 \) defined by the equations

\[ Z_0X_1 = Z_1X_0, \quad Z_0Y_2 = Z_2Y_0 \]

where \( Z_i, X_j, Y_k \) are respectively the homogeneous coordinates on \( CP^2, CP^1 \) and \( CP^1 \).

Let \( G \) be the automorphism group acting on \( CP^2 \times CP^1 \times CP^1 \) generated by \( \theta_j \) and permutations \( \tau (0 \leq i \leq 2) \)

\[ \theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_j e^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \]
for $\theta \in [0, 2\pi)$, and
\[
\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1].
\]

Let $\pi_0, \pi_1, \pi_2$ be the projection from $CP^2 \times CP^1 \times CP^1$ onto $CP^2, CP^1$ and $CP^1$. Define $\omega$ by
\[
\omega = \pi_0^* \omega_0 + \pi_1^* \omega_1 + \pi_2^* \omega_2
\]
\[
= \partial \overline{\partial} \log(\|Z_0\|^2 + \|Z_1\|^2 + \|Z_2\|^2) + \partial \overline{\partial} \log(\|X_0\|^2 + \|X_1\|^2) + \partial \overline{\partial} \log(\|Y_0\|^2 + \|Y_2\|^2)
\]
where $\omega_0, \omega_1, \omega_2$ are the Fubini-Study metrics $CP^2, CP^1$ and $CP^1$. By explicit calculation, it can be shown that $\omega|_M$ is in the first Chern class of $M$ (see [1]).

Consider the divisor
\[
\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}
\]
which defines a line bundle $(L, h)$ on $CP^2 \times CP^1 \times CP^1$, where
\[
h = \frac{1}{\|Z_0\|^2 + \|Z_1\|^2 + \|Z_2\|^2}(\|X_0\|^2 + \|X_1\|^2)(\|Y_0\|^2 + \|Y_2\|^2)
\]
then $(L, h)|_M \to M$ defines the anticanonical line bundle on $M$ whose curvature form $-\partial \overline{\partial} \log h$ gives the first Chern class on $M$.

Since $M \setminus \{\pi^{-1}(p_1) \cup \pi^{-1}(p_2)\}$ is isomorphic to $CP^2 \setminus \{p_1, p_2\}$, if we choose the inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ on $CP^2$, the Kähler metric
\[
\omega_{g_0} = \partial \overline{\partial} \log(1 + \|z_1\|^2 + \|z_2\|^2) + \partial \overline{\partial} \log(1 + \|z_1\|^2) + \partial \overline{\partial} \log(1 + \|z_2\|^2)
\]
can be extended to a Kähler metric $g_0$ on $M$ which belongs to $c_1(M)$. If we take different inhomogeneous coordinates $(w_0, w_1) = [w_0, w_1, 1]$, the corresponding Kähler metric is
\[
\omega_{g_1} = \partial \overline{\partial} \log(1 + \|w_0\|^2 + \|w_1\|^2) + \partial \overline{\partial} \log(1 + \|w_0\|^2) + \partial \overline{\partial} \log(\|w_0\|^2 + \|w_1\|^2)
\]
and we have
\[
\det g_0 = \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_1|^2)}
\]
\[
+ \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_2|^2)^2}
\]
\[
\det g_1 = \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(1 + |w_0|^2 + |w_1|^2)}
\]
\[
= \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2} + \frac{|w_0|^2}{(1 + |w_0|^2 + |w_1|^2)^2}.
\]

Consider the line bundle \((L^N, h_N) \to CP^2 \times CP^1 \times CP^1\). Then

\[
\dim H^0(CP^2 \times CP^1 \times CP^1, O(L^N)) = \frac{(N + 1)^3(N + 2)}{2}
\]

and \(\{Z_0^i Z_1^j Z_2^k X_0^{i_0} X_1^{i_1} Y_0^{j_0} Y_2^{j_2}\}_{i_0 + i_1 + j_0 + j_1 = k_0 + k_2 = N}\) is an orthogonal basis for \(H^0(CP^2 \times CP^1 \times CP^1, O(L^N))\).

Let \(M_1\) be the hypersurface of \(CP^2 \times CP^1 \times CP^1\) defined by the equations

\[Z_0 X_1 = Z_1 X_0\]

and \(M_2\) be the hypersurface of \(CP^2 \times CP^1 \times CP^1\) defined by the equations

\[Z_0 Y_2 = Z_2 Y_0.\]

Then \(M = M_1 \cap M_2\).

In view of the short exact sequences

\[
0 \to O(L^N - [M_1]) \to O(L^N) \to O(L^N|_{M_1}) \to 0
\]
\[
0 \to O(L^N|_{M_1} - [M]) \to O(L^N|_{M_1}) \to O(L^N|_{M}) \to 0
\]

we can choose \(N\) sufficiently large so that

\(H^1(CP^2 \times CP^1 \times CP^1, O(L^N - [M_1])) = H^1(M_1, O(L^N|_{M_1} - [M])) = 0.\)

Then \(H^0(CP^2 \times CP^1 \times CP^1, O(L^N)) \to H^0(M_1, O(L^N|_{M_1})) \to 0\)

\(H^0(M_1, O(L^N|_{M_1})) \to H^0(M, O(L^N|_{M})) \to 0\)

6
Hence \( \{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_1^{k_1} | M \}_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = N} \) contains an orthogonal basis for \( H^0(M, O(L^N|M)) \) and

\[
||Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_1^{k_1}||_{L^N_M}^2 = \frac{|Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_0^{k_0} Z_2^{k_2}|^2}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N}
\]
on \( CP^2 \setminus \{p_1, p_2\} \). By Corollary 2.1, for any \( \varphi \) in \( P_G(M, \omega_y) \), we have on \( CP^2 \setminus \{p_1, p_2\} \),

\[
\varphi([Z_0, Z_1, Z_2]) = \lim_{N \to \infty} \frac{1}{N} \log \sum_{i_0 + i_1 + i_2 = j_0 + j_1 + k_0 + k_2 = N} |a_{(N)}^{(\varphi)}| Z_0^{i_0 + j_0 + k_0} Z_1^{i_1 + j_1} Z_2^{i_2 + k_2} | \leq \text{Const.}
\]

Lemma 3.1 \( \frac{1}{n} \log \sum_{i_0 + i_1 + i_2 = j_0 + j_1 + k_0 + k_2 = N} |Z_0^{i_0 + j_0 + k_0} Z_1^{i_1 + j_1} Z_2^{i_2 + k_2}|^2 \leq \text{Const.} \)

Proof On the patch \( U_0 = \{Z_0 \neq 0\} \), let \( z_1 = \frac{Z_1}{Z_0} \) and \( z_2 = \frac{Z_2}{Z_0} \),

\[
\frac{1}{n} \log \sum_{i_0 + i_1 + i_2 = j_0 + j_1 + k_0 + k_2 = N} |Z_0^{i_0 + j_0 + k_0} Z_1^{i_1 + j_1} Z_2^{i_2 + k_2}|^2 \leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 + k_0 + k_2 = N} |Z_0^{i_0 + j_0 + k_0} Z_1^{i_1 + j_1} Z_2^{i_2 + k_2}|^2 \right) \leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 + k_0 + k_2 = N} |z_1^{i_1 + j_1} z_2^{i_2 + k_2}|^2 \right) \leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 + k_0 + k_2 = N} 1 \right) = \frac{1}{n} \log \left( \frac{(n + 1)^3 (n + 2)}{2} \right)
\]
On the patch $U_2 = \{Z_2 \neq 0\}$, let $w_0 = \frac{Z_0}{Z_2}$ and $w_1 = \frac{Z_1}{Z_2}$,

$$\frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2 \right) \leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|w_0^{i_0+j_0+k_0} w_1^{i_1+j_1}|^2}{(1 + |w_0|^2 + |w_1|^2)^n (1 + |w_0|^2 + |w_1|^2)^n} \right) \leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} 1 \right) = \frac{1}{n} \log \left( (n + 1)^3 (n + 2) \right) \leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2} \right) + \varepsilon \leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2} \right) + 2 \log \varepsilon + \varepsilon \leq \log \varepsilon + \text{const}
$$

This inequality holds for the patch $U_1 = \{Z_1 \neq 0\}$, and so the lemma is proved.

**Lemma 3.2** There exists $\varepsilon > 0$ such that for any $\varphi \in P_C(M, \omega_g)$ and $N$, there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0+i_1+i_2 = j_0+j_1 = k_0+k_2 = n$, and $(a_n^{(n)})_{i_0i_1i_2j_0j_1k_0k_2} > \varepsilon$.

**Proof** Otherwise, for any $\varepsilon > 0$, there exists $\varphi$ and $N$, such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0+i_1+i_2 = j_0+j_1 = k_0+k_2 = n$, we have $(a_n^{(n)})_{i_0i_1i_2j_0j_1k_0k_2} < \varepsilon$. By choosing $n$ large enough and with the lemma above, we have

$$\varphi([Z_0, Z_1, Z_2]) \leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2} \right) + \varepsilon \leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2} \right) + 2 \log \varepsilon + \varepsilon \leq \log \varepsilon + \text{const}
$$

Since $\varepsilon$ could be arbitrarily small, the above inequality would imply that $\varphi \rightarrow -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$. 
Proof of the theorem:

\[ \varphi([Z_0, Z_1, Z_2]) \]

\[ = \lim_{N \to \infty} \frac{1}{N} \log \left( \left( \frac{1}{(\sum_{i_0+j_0+k_0=0}^{i_0+j_0+k_0=N} 1) \left( \sum_{i_1+j_1+k_1=0}^{i_1+j_1+k_1=N} 1 \right) \left( \sum_{i_2+j_2+k_2=0}^{i_2+j_2+k_2=N} 1 \right) \right)} \right) \]

\[ \geq \frac{1}{N} \log \left( \left( \sum_{i_0+j_0+k_0=0}^{i_0+j_0+k_0=N} 1 \right) \left( \sum_{i_1+j_1+k_1=0}^{i_1+j_1+k_1=N} 1 \right) \left( \sum_{i_2+j_2+k_2=0}^{i_2+j_2+k_2=N} 1 \right) \right) \]

where \( i_0 + j_0 + k_0 = m, i_1 + j_1 + i_2 + k_2 = 3N - m. \)

On the patch \( U_0 = \{ Z_0 \neq 0 \}, \)

\[ \int_{U_0} e^{-\alpha \varphi} \omega^2_{g_0} \]

\[ \leq C_1 \int_{0<|z_1|,|z_2|<1} e^{-\alpha \log \left( \frac{|Z_0|^{\frac{2a}{N}}|Z_1|^{\frac{3-a}{N}}|Z_2|^{\frac{3-a}{N}}}{(\sum_{i_0+j_0+k_0=0}^{i_0+j_0+k_0=N} 1) \left( \sum_{i_1+j_1+k_1=0}^{i_1+j_1+k_1=N} 1 \right) \left( \sum_{i_2+j_2+k_2=0}^{i_2+j_2+k_2=N} 1 \right) \right)} \omega^2_{g_0} \]

\[ = C_1 \int_{0<|z_1|,|z_2|<1} \left( 1 + |z_1|^2 \right)^{\frac{2a}{N}} \left( 1 + |z_2|^2 \right)^{\frac{3-a}{N}} \left( 1 + |z_2|^2 \right)^{\frac{3-a}{N}} \omega^2_{g_0} \]

\[ \leq C_2 \int_{0<|z_1|,|z_2|<1} \frac{1}{|z_1|^{\frac{3a}{N}}} \frac{1}{|z_2|^{\frac{3a}{N}}} \omega^2_{g_0} \]

On the patch \( U_2 = \{ Z_2 \neq 0 \}, \)

\[ \int_{U_2} e^{-\alpha \varphi} \omega^2_{g_1} \]

\[ \leq C_4 \int_{0<|w_1|,|w_1|\leq 1} e^{-\alpha \log \left( \frac{|Z_0|^{\frac{2a}{N}}|Z_1|^{\frac{3-a}{N}}|Z_2|^{\frac{3-a}{N}}}{(\sum_{i_0+j_0+k_0=0}^{i_0+j_0+k_0=N} 1) \left( \sum_{i_1+j_1+k_1=0}^{i_1+j_1+k_1=N} 1 \right) \left( \sum_{i_2+j_2+k_2=0}^{i_2+j_2+k_2=N} 1 \right) \right)} \omega^2_{g_1} \]
\begin{align*}
    &= C_4 \int_{0<|w_0|,|w_1|\leq 1} (1 + |w_0|^2 + |w_1|^2)^\alpha (1 + |w_0|^2)^\alpha (|w_0|^2 + |w_1|^2)^\alpha |w_0|^{2\alpha m} |w_1|^{3\alpha - \frac{N\alpha}{2}} \omega_{g_s}^2 \\
    &\leq C_5 \int_{0<|w_0|,|w_1|\leq 1} (1 + |w_0|^2 + |w_1|^2)^\alpha (1 + |w_0|^2)^\alpha (|w_0|^2 + |w_1|^2)^\alpha |w_0|^{2\alpha m} |w_1|^{3\alpha - \frac{N\alpha}{2}} (|w_0|^2 + |w_1|^2) \omega_{g_s}^2 \\
    &\leq C_6 \int_{t=0}^1 \int_{s=0}^1 \frac{1}{s^{\frac{\alpha m}{N}} t^{\frac{2\alpha - \frac{N\alpha}{2}}{2N}} (s + t)^{1-\alpha}} ds dt \\
    &\leq C_6 \int_{s=0}^1 \frac{1}{s^{\frac{\alpha m}{N}} t^{\frac{2\alpha - \frac{N\alpha}{2}}{2N}} s^{(1-\alpha)p} (1-\alpha)q} ds dt \\
\end{align*}

where \((p + q = 1)\).

**Case 1:** If \(1 \leq \frac{m}{N} \leq 3\), then

\[
\frac{\alpha m}{N} + (1-\alpha)p < 1 \iff \alpha < \frac{1 - p}{3 - p} < \frac{1 - p}{\frac{m}{N} - p}
\]

\[
3\alpha - 1 < 1
\]

\[
\frac{3}{2} \alpha - \frac{\alpha m}{2N} + (1-\alpha)q < 1 \iff \alpha < \frac{1 - q}{\frac{3}{2} - \frac{m}{2N} - q}
\]

**Case 2:** If \(0 < \frac{m}{N} < 1\), then

\[
\frac{\alpha m}{N} + (1-\alpha)p < 1 \iff \alpha < 1
\]

\[
3\alpha - 1 < 1
\]

\[
\frac{3}{2} \alpha - \frac{\alpha m}{2N} + (1-\alpha)q < 1 \iff \alpha < \frac{1 - q}{\frac{3}{2} - q}
\]

So we could choose any \(\alpha < \frac{1}{3}\), which implies that \(\alpha_G(M, \omega) \geq \frac{1}{3}\).

Conversely, we choose

\[
\varphi_{\varepsilon} = \log \left( \frac{|Z_0|^6}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_1|^2)^{\varepsilon}} + \varepsilon \right) - \log(1 + \varepsilon)
\]

\(\in \mathcal{P}_G(M, \omega)\)
Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log \frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have

$$\lim_{\varepsilon \to 0} \int_M e^{-\alpha \varphi_\varepsilon} \omega^2 = \infty, \text{ for any } \alpha > \frac{1}{3}.$$ 

Hence we have shown $\alpha_G(M, \omega) = \frac{1}{3}$.

§4. Proof of Theorem 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold $M$ of dimension $n$ whose $\alpha(M)$ is greater than $\frac{n}{n+1}$. The following theorem is due to Tian and Zhu[10].

**Theorem 4.1** Let $(M, \omega)$ be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$, then there exist constants $\delta = \delta(n)$ and $C = C(n, \lambda_2(\omega) - 1) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have

$$F_{\omega}(\phi) \geq J_{\omega}(\phi)^{\delta} - C,$$

which is the same as

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_{\omega}(\phi)^{-\frac{1}{\delta}}} \int_M \phi \omega^n - J_{\omega}(\phi)^{\delta}.$$

This implies in particular the Moser-Trudinger inequality on $S^2$, which reads

$$\frac{1}{4\pi} \int_{S^2} e^{-\phi} \omega \leq e^{\frac{1}{4\pi} \int_{S^2} |\nabla \phi|^2 \omega - \frac{1}{4\pi} \int_{S^2} \phi}$$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \partial \bar{\partial} \phi$ and $\text{Ric}(\omega) = \omega + \partial \bar{\partial} h_\omega$.

Consider the Monge-Ampère equation

$$(\omega' + \partial \bar{\partial} \psi)^n = e^{h_\omega - t\psi} \omega^n$$

We will use the continuity method backwards and let $\phi_t$ be a smooth family which solve the above equation.

The following lemmas are well-known [9], but we add the proofs for the sake of completeness.
Lemma 4.1 \( \text{Ric}(\omega_t) \geq t \omega_t \), and we have equality if and only if \( t = 1 \).

**Proof**

\[
\text{Ric}(\omega_t) = -\partial \bar{\partial} \log \omega_t^n = -\partial \bar{\partial} \log \frac{\omega_t^n}{\omega^n} + \text{Ric}(\omega) = -\partial \bar{\partial} (h_\omega - t \phi_t) + \omega + \partial \bar{\partial} h_\omega
\]

\[
= \omega + t \phi_t = t \omega_t + (1 - t) \omega \geq t \omega_t.
\]

Lemma 4.2 For any \( \phi \in P(M, \omega) \), if the Green’s function of \( \omega' = \omega + \partial \bar{\partial} \phi \) is bounded from below, we have:

\[
-\inf_M \phi \leq \frac{1}{V} \int_M (-\phi) \omega' + C.
\]

**Proof** Since \( \omega + \partial \bar{\partial} \phi = \omega' \) and \( \omega' - \partial \bar{\partial} \phi > 0 \), we have \( \Delta_{\omega'} \phi \leq n \).

\[
-\phi = \frac{1}{V} \int_M (-\phi) \omega' + \frac{1}{V} \int_M \Delta_{\omega'} \phi(y) G_{\omega'}(x, y) \omega' + \frac{1}{V} \int_M n(G_{\omega'}(x, y) - \inf G_{\omega'}(x, y)) \omega' \]

\[
\leq \frac{1}{V} \int_M (-\phi) \omega' + \frac{1}{V} \int_M n \omega' \inf \omega' \phi,
\]

Let \((M, \omega)\) be a Kähler-Einstein manifold with \( \text{Ric}(\omega) = \omega \) and let \( P(M, \omega, K) = \{ \phi \in P(M, \omega) \mid G_{\omega + \partial \bar{\partial} \phi}(x, y) \geq -K \} \). Then we have:

**Proposition 4.1** Let \((M, \omega)\) be a Kähler-Einstein manifold with \( \text{Ric}(\omega) = \omega \). If \( \alpha(M) > \frac{n}{n+1} \), then there exist constants \( \delta(n, \alpha, K) \) and \( C(n, \alpha, \lambda_2(\omega) - 1, K) \) such that for any \( \phi \in P(M, \omega, K) \), we have

\[
F_{\omega}(\phi) \geq \delta J_{\omega}(\phi) - C.
\]

**Proof** Let \( \omega' = \omega + \partial \bar{\partial} \phi \), where \( \phi \in P(M, \omega, K) \).

\[
\frac{1}{V} \int_M e^{-\alpha \phi} \omega^n = \frac{1}{V} \int_M e^{-(\alpha_1 + \alpha_2 + \epsilon) \phi} \omega^n
\]

\[
\leq \frac{1}{V} \int_M e^{-(\alpha_1 + \alpha_2) \phi} \omega^n e^{-\epsilon \inf_M \phi},
\]

take \( p = \frac{1}{\alpha_1}, q = \frac{1}{1 - \alpha_1} \), we have
\[
\frac{1}{V} \int_M e^{-(\alpha_1 + \alpha_2)\phi} \omega^n \leq \frac{1}{V} \left( \int_M e^{-\alpha_1 p \phi} \omega^n \right)^{1/p} \left( \int_M e^{-\alpha_2 q \phi} \omega^n \right)^{1/q} \\
= \frac{1}{V} \left( \int_M e^{-\phi} \omega^n \right)^{\alpha_1} \left( \int_M e^{-\frac{\alpha_2}{1-\alpha_1} \phi} \omega^n \right)^{1-\alpha_1} \\
\leq C e^{\alpha_1 J_\omega(\phi) - \frac{\alpha_1}{p}} \int_M \phi \omega^n \left( \int_M e^{-\frac{\alpha_2}{1-\alpha_1} \phi} \omega^n \right)^{1-\alpha_1}
\]

by Lemma 4.2,

\[
e^{-\varepsilon} \inf_M \phi \leq e^{\varepsilon} \int_M (-\phi) \omega^n + C \\
= e^{\varepsilon J_\omega(\phi) - \frac{\varepsilon}{\alpha}} \int_M \phi \omega^n + C \\
\leq e^{\varepsilon (n+1) J_\omega(\phi) - \frac{\varepsilon}{\alpha}} \int_M \phi \omega^n + C.
\]

By Holder inequality,

\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq \left( \frac{1}{V} \int_M e^{-\alpha \phi} \omega^n \right)^{\frac{1}{\alpha}} \\
\leq C e^{\frac{\alpha_1 + (n+1) \varepsilon}{\alpha} J_\omega(\phi) - \frac{\alpha_1 + \varepsilon}{\alpha} \int M \phi \omega^n \left( \int_M e^{-\frac{\alpha_2}{1-\alpha_1} \phi} \omega^n \right)^{1-\alpha_1}} \\
\leq C e^{\frac{\alpha_1 + (n+1) \varepsilon}{\alpha} J_\omega(\phi) - \frac{\varepsilon}{\alpha} \int M \phi \omega^n \left( \int_M e^{-\frac{\alpha_2}{1-\alpha_1} (\phi - \sup \phi)} \omega^n \right)^{1-\alpha_1}}
\]

We need to determine \( \alpha_1, \alpha_2, \varepsilon \), which satisfy the following conditions

\[
\alpha = \alpha_1 + \alpha_2 + \varepsilon > 1 \\
\alpha > \alpha_1 + (n+1) \varepsilon \\
1 > \alpha_1
\]

So we will choose

\[
\alpha_2 = n\varepsilon + \varepsilon' \\
\alpha_1 = 1 - \alpha_2 - \varepsilon + \varepsilon'' = 1 - (n+1)\varepsilon - \varepsilon' + \varepsilon''
\]

where \( \varepsilon, \varepsilon', \varepsilon'' << 1 \), and \( \varepsilon' = o(\varepsilon), \varepsilon'' = o(\varepsilon') \).

Since \( \alpha(M) > \frac{n}{n+1} \), then we can choose \( \varepsilon, \varepsilon', \varepsilon'' \) small enough, then we have

\[
\frac{\alpha_2}{1-\alpha_1} = \frac{n\varepsilon + \varepsilon'}{(n+1)\varepsilon + \varepsilon' - \varepsilon''} < \alpha(M)
\]

13
and
\[ \int_M e^{-\frac{\alpha}{t-\alpha t_1}(\phi - \sup \phi)} \omega^n < \text{Const.} \]

Combined with the inequalities above, we have
\[ \frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)J_\omega(\phi) - \frac{1}{t} \int_M \phi \omega^n} \]

Which proves the lemma.

**Proof of Theorem 2**

We assume \( \omega \) is the Kähler-Einstein metric of \( M \). For any \( \phi \in P(M, \omega) \), put \( \omega' = \omega + \partial \bar{\partial} \phi \). Consider \( (\omega' + \partial \bar{\partial} \psi) = e^{h_{\omega'} + t \psi} \). By solving the Monge-Ampère equation backwards, we get the solutions \( \phi_t \), and \( \phi_1 = -\phi \).

For \( t > \frac{1}{2} \), let \( \omega_t = \omega' + \partial \bar{\partial} \phi_t = \omega + \partial \bar{\partial}(\phi_t - \phi_1) \), by Lemma 4.1,

\[ \text{Ric}(\omega_t) \geq \frac{1}{2} \omega_t. \]

which shows that the Green function of \( \omega_t \) is uniformly bounded from below.

Thus by proposition 4.1 and the calculation in [10] we have
\[ F_\omega(\phi_t - \phi_1) \geq \delta J_\omega(\phi_t - \phi_1) - C \]
\[ \geq C \text{osc}_M(\phi_t - \phi_1) - C_2 \]

and consequently,
\[ n(1 - t)J_{\omega}(\phi) = n(1 - t)J_{\omega'}(\phi_1) \geq (1 - t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) = F_\omega(\phi_t - \phi_1) \geq C \text{osc}_M(\phi_t - \phi_1) - C_2 \]

\[ F_\omega(\phi) = -F_{\omega'}(-\phi) \]
\[ = \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_1)) dt \geq (1 - t)(I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) \]
\[
\begin{align*}
\geq & \frac{1 - t}{n} J_{\omega'}(\phi_t) \\
\geq & \frac{1 - t}{n} J_{\omega'}(\phi_1) - 2(1 - t)(C_1 \text{osc}_{M}(\phi_t - \phi_1) - C_2) \\
\geq & \frac{1 - t}{n} J_{\omega}(\phi) - 2(1 - t)^2 nC_1 J_{\omega}(\phi) - C_3
\end{align*}
\]

The theorem follows by choosing \((1 - t) < \frac{1}{2n^2 C_1}\).

References


