

THE SZEGÖ KERNEL ON AN ORBIFOLD CIRCLE BUNDLE

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1 Introduction

The analysis of holomorphic sections of high powers L^N of holomorphic ample line bundles $L \rightarrow M$ over compact Kähler manifolds has been widely applied in complex geometry and mathematical physics. Any polarized Kähler metric g with respect to the ample line bundle L corresponds to the Ricci curvature of a hermitian metric h on L . Any orthonormal basis $\{S_0^N, \dots, S_{d_N}^N\}$ of $H^0(M, L^N)$ induces a holomorphic embedding Φ_N of M into CP^{d_N} . We call the pullback of the rescaled Fubini-Study metric $\frac{1}{N}\Phi_N^*g_{FS}$ the Bergman metric with respect to L^N . Tian[25] applied Hörmander's L^2 -estimate to produce peak sections and proves the C^2 convergence of the Bergman metrics. Zelditch[32] later generalized Tian's theorem by applying Boutet de Monvel-Sjöstrand[6] parametrix for the Szegö kernel. Namely

Theorem 1.1 (Zelditch[32]) *Let M be a compact complex manifold of dimension n and let $(L, h) \rightarrow M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each positive integer N , h induces a Hermitian metric h_N on L^N . Let $\{S_0^N, S_1^N, \dots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^N)$, $d_N + 1 = \dim H^0(M, L^N)$, with respect to the inner product:*

$$(s_1, s_2)_{h_N} = \int_M h_N(s_1(x), s_2(x)) dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g . Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 \sim a_0(z)N^n + a_1(z)N^{n-1} + a_2(z)N^{n-2} + \dots$$

for some smooth coefficients $a_j(z)$ with $a_0 = 1$. More precisely, for any k :

$$\left\| \sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 - \sum_{0 \leq j < R} a_j(z) N^{n-j} \right\|_{C^k} \leq C_{R,k} N^{n-R}$$

where $C_{R,k}$ depends on R, k and the manifold M .

In [17], Lu shows that each coefficient $a_j(z)$ is a polynomial of the curvature and its covariant derivatives and gives a method to compute them explicitly. In particular $a_1(z)$ is the scalar curvature with respect to g , which together with the asymptotic expansion helps Donaldson[11] prove that a metric of constant scalar curvature on a polarized Kähler manifold is the limit of balanced metrics.

Since orbifolds arise as degeneration limits of non-singular Kähler manifolds, the property of such limits are crucial to the understanding of the notion of K-stability conjectured by Tian to be equivalent to the existence of metrics of constant scalar curvature. Unfortunately the asymptotic expansions of the Bergman metrics fail near the singularities in the case of orbifolds. In this paper we generalize Zelditch's theorem to orbifolds of finite isolated singularities.

Theorem 1.2 *Suppose M is a compact Kähler orbifold of $\dim \geq 2$ with only finite isolated singularities $\{z_i\}_{i=1}^m$ and let $(L, h) \rightarrow M$ be a positive holomorphic orbifold line bundle. Let g be the orbifold Kähler metric on M corresponding to the Kähler form $\omega_g = \text{Ric}(h)$. For each N , h induces a hermitian metric h_N on L^N . Let $\{S_0^N, \dots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^N)$, where $d_N + 1 = \dim H^0(M, L^N)$, with respect to the inner product*

$$\langle s_1, s_2 \rangle_{h_N} = \int_M (s_1(z), s_2(z)) h^N dV_g.$$

Let $\{\delta_{z_i}(z)\}_{i=1}^m$ be the corresponding distributions. Then there exists an asymptotic expansion

$$\begin{aligned} \sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 &\sim a_0 N^n + a_1(z) N^{n-1} + \dots + a_n(z) + \sum_{i=1}^m b(i) \delta_{z_i}(z) \\ &\quad + a_{n+1}(z) N^{-1} + a_{n+2}(z) N^{-2} + \dots \end{aligned}$$

in the sense that

$$\sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 - (a_0 N^n + a_1(z) N^{n-1} + \dots + a_n(z))$$

weakly converges to $\sum_{i=1}^m b(i)\delta_{z_i}(z)$ as $N \rightarrow \infty$. Furthermore there exist constants $\delta > 0$ and $C_{R,k}$ such that

$$\left\| \sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 - \sum_{0 \leq j < R} a_j(z)N^{n-j} \right\|_{C^k} \leq C_{R,k}(N^{n-R} + N^{n+k/2}e^{-\delta Nr^2})$$

where r is the smallest geodesic distance from z to the singularities. In particular, $a_0 = 1$ and $a_1(z)$ is the scalar curvature of the orbifold (M, g) and $b(i) = \frac{1}{|G_i|} \sum_{1 \neq g \in G_i} \frac{1}{\det(I-g|T_{z_i})}$, where G_i is the structure group of z_i for $i = 1, \dots, m$.

We can define the embedding $\Phi_N: M \rightarrow \mathbf{CP}^{d_N}$ by sending $z \in M$ to

$$\Phi_N(z) = [S_0^N(z), \dots, S_{d_N}^N(z)] \in \mathbf{CP}^{d_N}$$

for N large enough and let ω_{FS} be the Fubini-Study metric on \mathbf{CP}^{d_N} .

Theorem 1.3 *Suppose M is a compact Kähler orbifold of $\dim \geq 2$ with only finite isolated singularities $\{z_i\}_{i=1}^m$ and let $(L, h) \rightarrow M$ be a positive holomorphic orbifold line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For any smooth plurisubharmonic function ϕ with $\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi + \omega \geq 0$, we denote \tilde{h} by $he^{-\phi}$ and $\omega_{\tilde{g}}$ by $Ric(\tilde{h})$. Let $\{\tilde{S}_0^N, \dots, \tilde{S}_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^N)$ with respect to the inner product*

$$\langle \tilde{s}_1, \tilde{s}_2 \rangle_{h_N} = \int_M (\tilde{s}_1(z), \tilde{s}_2(z)) \tilde{h}_N dV_{\tilde{g}}.$$

Then

$$\left\| \phi - \frac{1}{N} \log \left(\sum_{i=0}^{d_N} \|\tilde{S}_i^N(z)\|_{h_N}^2 \right) \right\|_{C^0(M)} \rightarrow 0.$$

Furthermore, if we assume that for each z_i its structure group G_i is abelian, then

$$\left\| \phi - \frac{1}{N} \log \left(\sum_{i=0}^{d_N} \|\tilde{S}_i^N(z)\|_{h_N}^2 \right) \right\|_{C^{1,\alpha}(M)} \rightarrow 0,$$

for any positive $\alpha < 1$. There also exist $\epsilon > 0$ and $N_0 > 0$ such that for all $N > N_0$,

$$\inf_{z \in M} \sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 \geq \epsilon N^n,$$

where $\{S_i^N\}_{i=0}^{d_N}$ is an orthonormal basis of $H^0(M, L^N)$ in Theorem 1.2.

Corollary 1.1 *With the same assumption in as Theorem 2.5, there exist constants $C_k > 0$ and $N_0 > 0$ such that for any $N > N_0$, we have*

$$\left\| \frac{1}{N} \Phi_N^* \omega_{FS} - \omega \right\|_{C^k} \leq C_k \left(\frac{1}{N} + N^{k/2} e^{-\delta N r^2} \right).$$

We conjecture that Theorem 1.3 should be true even without the assumption that the structure groups be abelian. Tian proved that any sequence of Kähler-Einstein surfaces with positive first Chern class converges to a Kähler-Einstein orbifold and that the singular points must be rational double points or of cyclic types. If our conjecture is true then there would exist a uniform constant $\epsilon > 0$ such that for any Kähler-Einstein surface (M, g) with $Ric(g) = g$ we would have

$$\inf_{x \in M} \sum_{i=0}^{d_N} \|S_i^N(z)\|_g^2(x) \geq \epsilon N^n.$$

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2 Orbifolds and orbifold vector bundles

We recall the definition of orbifolds which were introduced by Satake as V -manifolds [4, 20, 21].

Definition An *orbifold structure* on a Hausdorff separable topological space X is given by an open cover \mathbf{U} of X satisfying the following conditions:

1. Each $U \in \mathbf{U}$ has a local uniformization $\{\tilde{U}, G, \pi\}$ where \tilde{U} is a connected open neighborhood of the origin in \mathbf{C}^n and G is a finite group acting smoothly on \tilde{U} such that $U = \tilde{U}/G$ with π as the projection map. Let $\ker G_U$ the subgroup of G_U acting trivially on U .
2. If $V \subset U$, then there is a collection of injections $\{\tilde{V}, G_V, \pi_V\} \rightarrow \{\tilde{U}, G_U, \pi_U\}$. Namely, the inclusion $i : V \rightarrow U$ can be lifted to $\tilde{i} : \tilde{V} \rightarrow \tilde{U}$ and an injective homomorphism $i_{\#} : G_V \rightarrow G_U$ such that $i_{\#}$ is an isomorphism from $\ker G_V$ to $\ker G_U$ and \tilde{i} is $i_{\#}$ -equivariant.

3. For any point $x \in U_1 \cap U_2$, where $U_1, U_2 \in \mathbf{U}$, there is a $U_3 \in \mathbf{U}$ such that $x \in U_3 \subset U_1 \cap U_2$.

Definition An *orbifold bundle* B over an orbifold X with group Γ and fiber F consists of the following data:

1. For each local uniformization $\{\tilde{U}, G_U, \pi_U\}$ there is a bundle B_U over \tilde{U} with group Γ and fiber F together with an anti-isomorphism h_U of G_U into a group of bundle maps of B_U onto itself such that if b lies in the fiber over $\tilde{x} \in \tilde{U}$, then $h_U(g)b$ lies in the fiber over $g^{-1}\tilde{x}$ for $g \in G_U$.
2. For any $\tilde{i} : \tilde{V} \rightarrow \tilde{U}$ there is an induced $i_{\#}$ -equivariant bundle map $i^* : B_U \rightarrow B_V$.
3. If $\tilde{i}_1 : \tilde{V} \rightarrow \tilde{U}$ and $\tilde{i}_2 : \tilde{U} \rightarrow \tilde{W}$ then $(i_1 i_2)^* = i_2^* i_1^*$.

Suppose that $X = M/G$ and that $E \rightarrow M$ is a G -equivariant bundle then $E/G \rightarrow X$ is an orbifold vector bundle.

The tangent bundle T_X of X is defined by taking for T_U the tangent bundle over \tilde{U} , for $h_U(g)$ the inverse of the mapping of tangent vectors induced by g and i^* the inverse of the mapping of tangent vectors induced by i . If g is a metric on T_X then for each $\{\tilde{U}, G, \pi\}$, g_U is a G -invariant metric for (\tilde{U}) . We can also define the cotangent bundle T_X^* and $A^p(X)$ the bundle of differential p -forms over X in the same manner.

Definition Let $\{U_i\}_{i \in I}$ be a locally finite covering of X by open sets U_i such that $\{\tilde{U}_i, G_{U_i}, \pi_{U_i}\} \in \mathbf{U}$. By a smooth partition of unity for $\{U_i\}_{i \in I}$ we mean a collection of smooth functions $\{\psi_i\}$ such that $\text{supp}(\psi_i) \subset \tilde{U}_i$ and for each $x \in X$ $\sum_{i \in I} \psi_i(x) = 1$.

It is easy to show the existence of such partition of unity by shrinking each \tilde{U}_i a little so that we have a locally finite covering $\{V_i\}_{i \in I}$ with $V_i \subset U_i$. Then we can choose for each i a smooth function u_i on X such that $u_i = 1$ on \tilde{V}_i and $u_i = 0$ outside \tilde{U}_i . Then we can put $\psi_i = \frac{u_i}{\sum u_i}$.

Example 1

Let X be the quotient of \mathbf{CP}^1 by a cyclic group of order n defined by

$$[Z_0, Z_1] \sim [Z_0 e^{\frac{2k\pi}{n}}, Z_1]$$

for $k = 0, 1, 2, \dots, n - 1$. Then it is a "football" which has two isolated quotient singularities $[0, 1]$ and $[1, 0]$ with the cyclic structure group μ_n .

Example 2

Definition Let d_0, \dots, d_n be $n + 1$ positive integers. The weighted projective space P_{d_0, \dots, d_n} is a toric variety defined by

$$P_{d_0, \dots, d_n} = \{z \in \mathbf{C}^{n+1} - \{0\} \mid z \sim \lambda z, \lambda \in \mathbf{C}^*\}$$

where \mathbf{C}^* acts by

$$\lambda(Z_0, \dots, Z_n) = (\lambda^{d_0} Z_0, \dots, \lambda^{d_n} Z_n).$$

As for the case of projective spaces, we let $U_i = \{Z_i \neq 0\}$, then

$$U_i = \left\{ \left(\frac{Z_0}{Z_i^{d_0/d_i}}, \dots, \frac{Z_i}{Z_i}, \dots, \frac{Z_n}{Z_i^{d_n/d_i}} \right) \right\}$$

which is exactly \mathbf{C}^n / μ_{d_i} if $d = \gcd(d_0, \dots, d_n) = 1$.

We have the following properties of weighted projective space:

1. The above C^* -action is free if and only if $d_i = d_j$, for all $i, j = 0, \dots, n$.
2. Let $d = \gcd(d_0, \dots, d_n)$ be the greatest common divisor of d_0, \dots, d_n , then P_{d_0, \dots, d_n} is homeomorphic to $P_{d_0/d, \dots, d_n/d}$.
3. Weighted projective spaces are orbifolds which have singularities with cyclic structure groups acting diagonally. In particular if $(d_i, d_j) = 1$ for all $i \neq j$, $i, j = 0, \dots, n$, then P_{d_0, \dots, d_n} has only isolated singularities.

Now we will state the Riemann-Roch-Kawasaki Theorem which enables us to determine the coefficients of the currents in the expansion in Theorem 1.2.

For each local uniformization $\{\tilde{U}, G_U\}$ and each $g \in G_U$, we consider \tilde{U}^g as a complex manifold on which the centralizer $Z_{G_U}(g)$ acts. For $V \subset U$, the open embedding $i : \tilde{V} \rightarrow \tilde{U}$ defines a natural open embedding $\tilde{V}^h / Z_{G_V}(h) \rightarrow \tilde{U}^g / Z_{G_U}(g)$ of analytic spaces, where $g = i^\#(h)$. We patch all the $\tilde{U}^g / Z_{G_U}(g)$ together by such identification which gives a disjoint union of complex orbifolds of various dimensions:

$$X \amalg \tilde{\Sigma} X = \cup_{\{\tilde{U}, G_U\}, g \in G_U} \tilde{U}^g / Z_{G_U}(g).$$

We have a canonical map $\tilde{\Sigma} X \rightarrow X$ covered locally by the inclusion $\tilde{U}^g \subset \tilde{U}$. For each $x \in X$ we can choose a local uniformization $\{\tilde{U}_x, G_x\}$ such that $x \in \tilde{U}_x$ is a

fixed point of G_x . G_x is unique up to isomorphism. Then the number of pieces of $\tilde{\Sigma}X$ is equal to the number of the conjugacy classes of G_x other than the identity class.

Let $\tilde{\Sigma}X_1, \dots, \tilde{\Sigma}X_k$ be all the connected components of $\tilde{\Sigma}X$. We define m_i for each $\tilde{\Sigma}X_i$ by

$$m_i = |\ker[Z_{G_U}(g) \rightarrow \text{Aut}(\tilde{U}^g)]|.$$

Let $\sum_{g \in G_U} \mathcal{L}^g(U; E_U)$ be the equivariant Todd form on $X \amalg \tilde{\Sigma}X$ which represents a cohomology class $\mathcal{L}(X; E) + \mathcal{L}^\Sigma(X; E)$ in $H^*(X \amalg \tilde{\Sigma}X; C)$. Then we have the following Riemann-Roch-Kawasaki theorem.

Theorem 2.1 [16] *Let X be a compact complex orbifold and let $E \rightarrow X$ be a holomorphic orbifold vector bundle. Then we have*

$$\chi(X; \mathcal{O}_X(E)) = \langle \mathcal{L}(X; E), [X] \rangle + \sum \frac{1}{m_i} \langle \mathcal{L}^\Sigma(X; E), [\tilde{\Sigma}X_i] \rangle.$$

In particular, if X only has isolated singularities $\{x_j\}_{j=1, \dots, m}$, we have

$$\chi(X; \mathcal{O}_X(E)) = \langle \mathcal{L}(X; E), [X] \rangle + \sum_{i=1}^m \sum_{1 \neq g \in G_{x_i}} \frac{1}{\det(1 - g|T_{x_i})}.$$

Notice that since $g|T_{x_j}$ is orthogonal and has no eigenvalue of 1, it follows that $\det(1 - g|T_{x_j}) > 0$.

3 The $\bar{\partial}_b$ -equation

In this section we will establish the $\bar{\partial}_b$ -equation for orbifolds and obtain subelliptic estimates which gives the Hodge decomposition for $\bar{\partial}_b$ operator. We will essentially follow Folland and Kohn [12].

Definition Let X be a compact, orientable real orbifold of dimension $2n - 1$. A partially complex structure on X is an $(n - 1)$ -dimensional orbifold subbundle S of CTX such that

1. $S \cap \bar{S} = \{0\}$,
2. if L, L' are local sections of S then so is $[L, L']$.

Definition If X is partially complex, we define the orbifold vector bundle $B^{p,q}$ ($0 \leq p, q \leq n-1$) by $B^{p,q} = \Lambda^p S^* \otimes \Lambda^q \bar{S}^*$, which we can identify with an orbifold subbundle of $\Lambda^{p+q} CTX^*$. We denote by $\mathcal{B}^{p,q}$ the space of smooth sections of $B^{p,q}$, and we define $\bar{\partial}_b : \mathcal{B}^{p,q} \rightarrow \mathcal{B}^{p,q+1}$ as follows:

1. If $\phi \in \mathcal{B}^{p,0}$, then $\bar{\partial}_b \phi$ is defined by

$$\langle \bar{\partial}_b \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes V \rangle = V \langle \phi, L_1 \Lambda \dots \Lambda L_p \rangle$$

for all sections L_1, \dots, L_p of S and V of \bar{S} .

2. If $\phi \in \mathcal{B}^{p,q}$,

$$\begin{aligned} & (q+1) \langle \bar{\partial}_b \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes (V_1 \Lambda \dots \Lambda V_{q+1}) \rangle \\ &= \sum_{j=1}^{q+1} (-1)^{j+1} V_j \langle \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes (V_1 \Lambda \dots \hat{V}_j \dots \Lambda V_{q+1}) \rangle \\ & \quad + \sum_{i < j} (-1)^{i+j} \langle \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes ([V_i, V_j] \Lambda V_1 \Lambda \dots \hat{V}_i \dots \hat{V}_j \dots \Lambda V_{q+1}) \rangle. \end{aligned}$$

$\bar{\partial}_b \phi$ is well-defined since $\bar{\partial}_b$ commutes with G .

$$\begin{aligned} & (q+1) \langle g^* \bar{\partial}_b \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes (V_1 \Lambda \dots \Lambda V_{q+1}) \rangle |_{\tilde{x}} \\ &= (q+1) \langle \bar{\partial}_b \phi, g_* (L_1 \Lambda \dots \Lambda L_p) \otimes g_* (V_1 \Lambda \dots \Lambda V_{q+1}) \rangle |_{g\tilde{x}} \\ &= \sum_{j=1}^{q+1} (-1)^{j+1} V_j \langle g^* \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes (V_1 \Lambda \dots \hat{V}_j \dots \Lambda V_{q+1}) \rangle |_{\tilde{x}} \\ & \quad + \sum_{i < j} (-1)^{i+j} \langle g^* \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes ([V_i, V_j] \Lambda V_1 \Lambda \dots \hat{V}_i \dots \hat{V}_j \dots \Lambda V_{q+1}) \rangle |_{\tilde{x}} \\ &= (q+1) \langle \bar{\partial}_b g^* \phi, (L_1 \Lambda \dots \Lambda L_p) \otimes (V_1 \Lambda \dots \Lambda V_{q+1}) \rangle |_{\tilde{x}}. \end{aligned}$$

Let L_1, \dots, L_{n-1} be a local basis for sections of S over \tilde{U} , so $\bar{L}_1, \dots, \bar{L}_{n-1}$ is a local basis for sections of \bar{S} . We choose a local section N of CTX such that $L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, N$ span $CTX|_U$ and we may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \bar{L}_k + c_{ij} N$$

is hermitian and it is called Levi form.

Proposition 3.1 *The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point \tilde{x} are independent of the choice of L_1, \dots, L_{n-1}, N .*

We say that X satisfies condition $Y(q)$ if the Levi form have $\max(q + 1, n - q)$ eigenvalues of the same sign or $\min(q + 1, n - q)$ pairs of eigenvalues with opposite signs at each point. Notice that for $n = 2$ and $q = 1$ the condition $Y(q)$ is never satisfied and if X is pseudoconvex then $Y(q)$ is satisfied for $n > 2$ and $0 < q < n - 1$.

We will now define the Sobolev norms on orbifolds.

Definition Let X be an n -dimensional compact orbifold. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a locally finite covering of X with their uniformization $\{\tilde{U}_\alpha\}$ and coordinate mappings $\varphi_\alpha : \tilde{U}_\alpha \rightarrow R^n$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then for $s \in R$ and any k -form ϕ over X we define $\|\phi\|_s^2 = \sum_\alpha \|(\psi_\alpha \phi) \circ \varphi_\alpha^{-1}\|_s^2$. The norm $\|\cdot\|_s$ is not intrinsic, but it is independent of the choice of local coordinate charts, partition of unity and the coordinate mappings up to equivalence.

We can choose a hermitian metric on CTX such that S, \bar{S} and N are orthogonal to each other and we can then assume that $L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, N$ are orthonormal. We can define the Sobolev spaces $H_s^{p,q}$ for all real s by completing $\mathcal{B}^{p,q}$ appropriately and define the adjoint operator $\bar{\partial}_b^*$ and the Laplacian $\Delta_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$. If $\omega_1, \dots, \omega_{n-1}, \bar{\omega}_1, \dots, \bar{\omega}_{n-1}, \eta$ is the dual basis to $L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, N$, we write $\phi \in \mathcal{B}^{p,q}$ as $\phi = \sum_{IJ} \phi_{IJ} \omega^I \bar{\omega}^J$ and

$$\begin{aligned} \bar{\partial}_b \phi &= (-1)^p \sum_{kIJK} \delta_{kJ}^K \bar{L}_k(\phi_{IJ}) \omega^I \wedge \bar{\omega}^K + \text{terms of order zero,} \\ \bar{\partial}_b^* \phi &= (-1)^{p+1} \sum_{kIHK} \delta_{kH}^J \bar{L}_k(\phi_{IJ}) \omega^I \wedge \bar{\omega}^H + \text{terms of order zero.} \end{aligned}$$

We define the hermitian form Q_b on $\mathcal{B}^{p,q}$ by

$$\begin{aligned} Q_b(\phi, \psi) &= (\bar{\partial}_b \phi, \bar{\partial}_b \psi) + (\bar{\partial}_b^* \phi, \bar{\partial}_b^* \psi) + (\phi, \psi) \\ &= ((\Delta_b + I)\phi, \psi). \end{aligned}$$

Lemma 3.1 *If X satisfies condition $Y(q)$, then for all $\phi \in \mathcal{B}^{p,q}$ with support in U we have $\|\phi\|_{1/2}^2 \leq C Q_b(\phi, \phi)$.*

This is a local subelliptic estimate which can be proved in the same way for non-singular partial complex manifolds with the condition $Y(Q)$ satisfied. See [12].

Theorem 3.1 *If X satisfies condition $Y(q)$, then for all $\phi \in \mathcal{B}^{p,q}$ we have $\|\phi\|_{1/2}^2 \leq C Q_b(\phi, \phi)$.*

Proof We simply apply the partition of unity and

$$\|\phi\|_{1/2}^2 \leq \sum C_1 \|\psi_i \phi\|_{1/2}^2 \leq C_2 \sum Q_b(\psi_i \phi, \psi_i \phi) \leq C_3 Q_b(\phi, \phi).$$

We denote the harmonic space by $\mathcal{H}_b^{p,q} = \{\phi \in \mathcal{B}^{p,q} : \Delta_b \phi = 0\}$ which is finite dimensional. We then have the following Hodge decomposition

$$H_0^{p,q} = \bar{\partial}_b \bar{\partial}_b^* \text{Dom}(\Delta_b) \oplus \bar{\partial}_b \bar{\partial}_b^* \text{Dom}(\Delta_b) \oplus \mathcal{H}_b^{p,q}.$$

Let H_b be the orthogonal projection on $\mathcal{H}_b^{p,q}$ and G_b be the inverse of Δ_b on $(\mathcal{H}_b^{p,q})^\perp$ and zero on $\mathcal{H}_b^{p,q}$. By the same argument in [12] we have the following theorem as in the smooth case.

Theorem 3.2 *Suppose X satisfies condition $Y(q)$ then:*

1. G_b is a compact operator.
2. For any $a \in H_0^{p,q}$, $a = \bar{\partial}_b \bar{\partial}_b^* a + \bar{\partial}_b^* \bar{\partial}_b a + H_b a$.
3. $G_b H_b = H_b G_b = 0$; $G_b \Delta_b = \Delta_b G_b = I - H_b$ on $\text{Dom}(\Delta_b)$; and if G_b is also defined on $H_0^{p,q+1}(H_0^{p,q-1})$, $G_b \bar{\partial}_b = \bar{\partial}_b G_b$ on $\text{Dom}(\bar{\partial}_b)$ ($G_b \bar{\partial}_b^* = \bar{\partial}_b^* G_b$ on $\text{Dom}(\bar{\partial}_b^*)$).
4. $G_b \mathcal{B}^{p,q} \subset \mathcal{B}^{p,q}$ and $\|G_b a\|_s \leq C \|a\|_{s-1}$ holds uniformly for $a \in \mathcal{B}^{p,q}$ for each positive integer s .

Corollary 3.1 *If $X = \partial M$ is pseudoconvex, then the Szegö projector on X is given by*

$$S = I - \bar{\partial}_b^* G_b \bar{\partial}_b.$$

Proof For $n > 2$ and $q = 1$ X satisfies the condition $Y(1)$ since X is pseudoconvex and we have by the previous theorem that

$$I = \bar{\partial}_b \bar{\partial}_b^* G_b + \bar{\partial}_b^* \bar{\partial}_b G_b + H_b.$$

So $\bar{\partial}_b f = \bar{\partial}_b \bar{\partial}_b^* G_b \bar{\partial}_b f$ where G_b is the Green operator. Thus $\bar{\partial}_b \bar{\partial}_b^* G_b$ is the orthogonal projector onto $\text{Im} \bar{\partial}_b$ and $\bar{\partial}_b^* G_b \bar{\partial}_b$ is the orthogonal projector onto $(\text{Ker} \bar{\partial}_b)^\perp$ in $L^2(X)$. Therefore the Szegö projector S which is the orthogonal projection from $L^2(X)$ to $\text{ker}(\bar{\partial}_b) \cap L^2(X)$ can be written as

$$S = I - \bar{\partial}_b^* G_b \bar{\partial}_b.$$

4 From line bundle to circle bundle

Let $\mathcal{O}(1) \rightarrow \mathbf{CP}^n$ be the hyperplane line bundle and let \langle, \rangle be its natural hermitian metric. Let $M \in \mathbf{CP}^n$ be a projective manifold and let L be the restriction of $\mathcal{O}(1)$ to M and h be the restriction of \langle, \rangle to L . The following lemma is due to Grauert.

Lemma 4.1 *Let $D = \{(z, v) \in L^* : h(v, v) \leq 1\}$. Then D is a strictly pseudoconvex domain in L .*

Here L^* is the dual line bundle to L . The boundary of D is a principal S^1 bundle $X \rightarrow M$ defined by $\rho : L^* \rightarrow \mathbb{R}$, $\rho(z, v) = 1 - |v|_z^2$, where $v \in L_z^*$ and $|v|_z$ is its norm in the metric induced by h . $D = \{\rho > 0\}$. We will denote the S^1 action by $r_\theta x$ and its infinitesimal generator by $\frac{\partial}{\partial \theta}$ and ρ is S^1 -invariant.

Now replace M by an Kähler orbifold on which there is an orbifold line bundle $L \rightarrow M$ equipped with the orbifold hermitian metric h .

Lemma 4.2 *Let $D = \{(z, v) \in L^* : h(v, v) \leq 1\}$ and $X = \partial D$. Then both D and X have an orbifold structure. Furthermore, X is an orbifold circle bundle over M .*

The defining function $\rho(z, v) = 1 - |v|^2$ is globally defined on the orbifold L^* . Since $-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h^* = Ric(h^*) < 0$ we have $\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h^* > 0$. Therefore on each small enough local uniformization ρ is convex with respect to a choice of trivialization of L^* and coordinates on M since $Ric(h) > 0$.

Proposition 4.1 *There exists a smooth function $\psi(x, y)$ on each local uniformization \tilde{U} such that*

1. $\psi(x, x) = \frac{1}{i} \rho(x)$
2. $d_x'' \psi(x, y)$ and $d_y' \psi(x, y)$ vanish on the diagonal $\{x = y\}$ to infinite order.
3. $\psi(x, y) = -\overline{\psi(y, x)}$.

By making $\tilde{\psi}(x, y) = \frac{1}{|G|} \sum_{g \in G} \psi(gx, gy)$ and since $\rho(x)$ is G -invariant we can assume $\psi(\cdot, \cdot)$ is invariant under the diagonal action by G .

Proposition 4.2 *There exists a constant $C > 0$ such that*

$$Im \psi(x, y) \geq C(d(x, X) + d(y, X) + |x - y|^2) + O(|x - y|^3)$$

Proof

$$\psi(x+h, x+k) \sim \frac{1}{i} \sum \frac{\partial^{\alpha+\beta} \rho}{\partial z^\alpha \partial \bar{z}^\beta}(x) \frac{h^\alpha \bar{k}^\beta}{\alpha! \beta!}.$$

$$\frac{1}{i} [\psi(x, y) + \psi(y, x) - \psi(x, x) - \psi(y, y)] = L_\rho(x-y) + O(|x-y|^3).$$

Also

$$Im\psi(x, y) = \frac{1}{2i} (\psi(x, y) - \overline{\psi(x, y)}) = \frac{1}{2i} (\psi(x, y) + \psi(y, x)).$$

Notice ψ is only locally defined and in general ψ cannot be globally defined as in the smooth case in [6] due to the cancelling of the group action. Also we can always assume $Im\psi \geq 0$ by shrinking the uniformization a little.

Denote by $T'D, T''D \subset TD \otimes C$ the holomorphic and antiholomorphic subspaces, and define $d'f = df|_{T'}$ and $d''f = df|_{T''}$ for $f \in C^\infty(D)$. Then X inherits an orbifold CR structure $CTX = T' \oplus T'' \oplus C(\frac{\partial}{\partial \theta})$. Denote by $T'X$ the space of holomorphic vector fields on D which are tangent to X . They are given in local coordinates by vector fields $\sum a_j \frac{\partial}{\partial \bar{z}_j}$ such that $\sum a_j \frac{\partial}{\partial \bar{z}_j} \rho = 0$. A local basis is given by the vector fields $Z_j^k = \frac{\partial}{\partial \bar{z}_j} - (\frac{\partial \rho}{\partial \bar{z}_k})^{-1} (\frac{\partial \rho}{\partial \bar{z}_j}) \frac{\partial}{\partial \bar{z}_k}$ for $j \neq k$.

The Cauchy-Riemann operator on X is defined by

$$\bar{\partial}_b : C^\infty(X) \rightarrow C^\infty(X, (T'')^*)$$

$$\bar{\partial}_b f = df|_{T''}.$$

It's easy to see $T', T'', \partial\theta$ and $\bar{\partial}_b$ coincide with S, \bar{S}, N and $\bar{\partial}_b$ in the previous section.

Lemma 4.3 $[D_j^k, D_m^k] = 0$.

Proof It can be shown by straightforward calculation.

Lemma 4.4 *The characteristic cone Σ of $\bar{\partial}_b$ is the real cone of T^*X orthogonal to T'' and is generated by*

$$\frac{1}{i} d' \rho|_X = -\frac{1}{i} d'' \rho|_X.$$

Lemma 4.5 $\sigma([Z, W^*]) = L_\rho(Z, W)$ where Z and W are two C^∞ local sections of T'' .

Proof If $Z = \sum a_j \frac{\partial}{\partial \bar{z}_j}$, $W = \sum b_j \frac{\partial}{\partial \bar{z}_j}$, then

$$\begin{aligned} \sigma([Z, W^*])\left(\frac{1}{i}d'\rho\right) &= -\sigma([Z, \bar{W}])\left(\frac{1}{i}d'\rho\right) \\ &= -\langle [Z, \bar{W}], \frac{1}{i}d'\rho \rangle = -\langle \sum a_j \frac{\partial \bar{b}_k}{\partial z_j} \frac{\partial}{\partial z_k}, d'\rho \rangle|_X \\ &= -\sum a_j \frac{\partial \bar{b}_k}{\partial z_j} \frac{\partial \rho}{\partial z_k}. \end{aligned}$$

Since $\langle \bar{W}, d'\rho \rangle = \langle \bar{W}, d\rho \rangle = 0$ on X and Z is tangent to X we have

$$L_Z \langle W, d'\rho \rangle = a_j \left(\frac{\partial \bar{b}_k}{\partial z_j} \frac{\partial \rho}{\partial z_k} + \bar{b}_k \frac{\partial^2 \rho}{\partial z_j \partial z_k} \right) = 0.$$

Therefore

$$\sigma([Z, W^*])\left(\frac{1}{i}d'\rho\right) = L_\rho(Z, W).$$

The Hardy space $H^2(X)$ is the space of boundary values of holomorphic functions on D which are in $L^2(X)$, i.e. $H^2 = \ker \bar{\partial}_b \cap L^2(X)$. The S^1 action commutes with $\bar{\partial}_b$, hence $H^2(X) = \oplus_N H_N^2(X)$, where $H_N^2(X) = \{f \in H^2(X) : f(\tau_\theta x) = e^{iN\theta} f(x)\}$.

A section of L determines an equivariant function \hat{s} on $L^* - \{0\}$ by $\hat{s}(z, \lambda) = \langle \lambda, s(z) \rangle$ where $z \in M$ and $\lambda \in L_z^*$. Similarly, a section s_N of L^N determines an equivariant function \hat{s}_N on $(L^N)^* - 0$ by $\hat{s}_N(z, \lambda) = \langle \lambda^N, s_N(z) \rangle$.

Lemma 4.6 *The map $s \rightarrow \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $H^2(X)$.*

We can generalize the above statements to orbifolds and holomorphic orbifold line bundles without difficulty.

We denote by $\Pi : L^2(X) \rightarrow H^2(X)$ and $\Pi_N : L^2(X) \rightarrow H_N^2(X)$ respectively the orthogonal projections. Their kernels are defined by

$$\begin{aligned} \Pi f(x) &= \int_X \Pi(x, y) f(y) d\mu(y) \\ \Pi_N f(x) &= \int_X \Pi_N(x, y) f(y) d\mu(y). \end{aligned}$$

Let $\{S_i^N\}$ be an orthonormal basis of $H^0(M, L^N)$.

Proposition 4.3 $\|S_j^N(z)\|_{h_N}^2 = |\hat{S}_j^N(x)|^2$ for any x with $\pi(x) = z$.

Proof Let e_L be a local G -invariant holomorphic section e_L of L over a local uniformization $\{\tilde{U}, G\}$. It induces sections e_L^N of $L^N|_{\tilde{U}}$ and let $S_i^N(z) = f_i^N(z)e_L^N(z)$ for a holomorphic function f_i^N on U . Then

$$\hat{S}_i^N(z, u) = \langle u^N, S_i^N(z) \rangle = f_i^N(z) \langle u^N, e_L^N(z) \rangle = f_i^N(z) a^{\frac{N}{2}} \langle u, \frac{e_L}{|e_L|}(z) \rangle^N.$$

So we have

$$\hat{S}_i^N(z, \theta) = f_i^N a(z)^{N/2} e^{iN\theta}.$$

Hence $|\hat{S}_i^N(z, \theta)|^2 = a(z)^N |f_i^N(z)|^2 = \|S_i^N(z)\|_{h_N}^2$.

Proposition 4.4 $\{\hat{S}_i^N\}$ is an orthonormal basis of $H^2(X)$.

Proof Let $dV_g = \frac{\omega_g^n}{n!}$ be the volume form of (M, g) . Then we have

$$\begin{aligned} \langle S_i^N, S_j^N \rangle &= \int_M h_N(S_i^N, S_j^N) dV_g \\ &= \int_M a^N(z) f_i^N(z) \overline{f_j^N(z)} dV_g \\ &= \int_X \hat{S}_i^N \overline{\hat{S}_j^N} d\mu, \end{aligned}$$

where $d\mu = \alpha \wedge d\alpha^n/n! = d\theta \wedge \pi^* \omega_g^n$ is the G_U -invariant volume form on any local uniformization \tilde{U} .

5 The local model

In this and the following section we will follow the method by L. Boutet de Monvel and J. Sjöstrand in [6] with a little modification near the singularities to prove a similar formula for the Szegö kernel for pseudoconvex domain with quotient singularities.

Let $(x, y) \in R^n = R^p \times R^q$ and (ξ, η) be the dual variable. Let Σ be the cone $\{x = \xi = 0\}$. Let D be a system of pseudo-differential operators $D_0 = \frac{1}{i}(\frac{\partial}{\partial x_j} + x_j |D_y|)$, $j = 1, \dots, p$. Let R be a linear continuous operator: $C_0^\infty(R^q) \rightarrow C^\infty(R^n)$ defined by

$$Rf(x, y) = (2\pi)^{-q} \int e^{iy\eta - \frac{1}{2}|x|^2|\eta|} (|\eta|/\pi)^{p/4} \hat{f}(\eta) d\eta.$$

One has $D_0R = 0$, $R^*R = I$ and

$$I \sim RR^* + L_0D_0.$$

where $L_0 \in OPS^{-1,-1}(R^n, \Sigma)$.

The operator RR^* is defined by the oscillatory integral

$$RR^*f(x, y) = (2\pi)^{-q} \int \int e^{i\langle y-y', \eta \rangle + \frac{1}{2}(|x|^2 + |x'|^2)|\eta|} (|\eta|/\pi)^{p/2} f(x', y') dx' dy' d\eta.$$

The phase function is defined by

$$\phi = \langle y - y', \eta \rangle + \frac{1}{2}(|x|^2 + |x'|^2)|\eta|.$$

Let Σ^0 be the cone defined by the complex equation

$$\sigma(D_0) = 0.$$

Then $\Sigma^0 \cap \overline{\Sigma^0}$ is the complexified cone of the real cone $\{x = \xi = 0\}$. The canonical relation C_0^+ is the complex cone satisfying

$$\begin{aligned} y' &= y + i/2(|x|^2 + |x'|^2) \\ \xi &= ix|\eta| \\ \xi' &= -ix'|\eta| \\ \eta' &= \eta. \end{aligned}$$

It is easy to see that C_0^+ is contained in $\Sigma^0 \times \overline{\Sigma^0}$ and contains $\text{diag}(\Sigma \times \Sigma)$.

Proposition 5.1 *The canonical relation C_0^+ is unique and satisfies*

1. $C_0^+ \subset \Sigma^0 \times \overline{\Sigma^0}$
2. *The set of all real points of C^+ is exactly the diagonal of $\Sigma \times \Sigma$*
3. C_0^+ *is positive.*

6 The Szegő kernel on X

Proposition 6.1 [5] *Let $\{Z_j\}$, $j = 1, \dots, v$ be homogeneous pseudo-differential operators of degree m and Σ the characteristic cone. If $\{Z_j\}$ satisfy*

1. $\sigma\{[Z_j^*, Z_k]\}$ is positive definite on Σ
2. $[Z_j, Z_k] \sim \sum A_{jk}^i Z_i$,

then there exists an elliptic Fourier integral operator transforming the left ideal generated by the Z_j into the ideal generated by the $\frac{1}{i}(\frac{\partial}{\partial x_j} + ix_j|D_y|)$, $j = 1, \dots, v$.

Therefore, for any $\xi \in \Sigma$, there exists a canonical isomorphism Φ defined on a neighborhood of ξ and an elliptic Fourier integral operator V associated with Φ on a neighborhood of ξ such that $\bar{\partial}_b = V^{-1}CD_0V$ on a neighborhood of ξ , where C is a matrix of elliptic pseudo-differential operators.

Proposition 6.2 *There exists one canonical relation C^+ which is almost analytic on $T^*\tilde{U} \times T^*\tilde{U}$, unique up to equivalence and satisfies*

1. $C^+ \subset \Sigma^0 \times \overline{\Sigma^0}$.
2. *The set of all real points of C^+ is exactly the diagonal of $\Sigma \times \Sigma$.*
3. C^+ is positive.

Proposition 6.3 C^+ is the canonical relation associated with the phase function $t\psi(x, y)$ on $\tilde{U} \times \tilde{U} \times R_+$.

Proof Observe that

1. $t\psi$ has no critical points
2. $\frac{\partial}{\partial t}t\psi = \psi = 0$ implies that the real critical points is $\text{diag}(X) \times R_+$ and on $\text{diag}(X)$ $d_x\psi = -d_y\psi = \frac{1}{i}d'\rho|X \neq 0$
3. $\text{Im}\psi \geq 0$
4. The set of real points of C^+ is exactly $\text{diag}\Sigma^+$ as above.

5. Since $d_x''\psi$ and $d_y'\psi$ vanish to infinite order on diagonal, C^+ is contained in $\Sigma^0 \times \overline{\Sigma^0}$.

Proposition 6.4 *For each point x of X there exists a neighborhood of x with its local uniformization \tilde{U} and regular operators S and L such that*

1. $S \sim S^* \sim S^2$
2. $\bar{\partial}_b S \sim 0$ and $I \sim L\bar{\partial}_b + S$,

where S is uniquely determined up to an operator of degree $-\infty$ and $S \in I_c^0(\tilde{U}^2, C^+)$.

Proof If A and B are two operators defined on distribution on \tilde{U} . then by $A \sim B$ we mean $R = A - B$ is an operator of order $-\infty$, or the kernel distribution of R is C^∞ on $\tilde{U} \times \tilde{U}$.

Uniqueness: Suppose there exist S' and L' which also satisfy conditions 1 and 2. The assertion of the theorem is local. Suppose W is an open cone of $T^*\tilde{U} - 0$, then on W

$$S \sim (S' + L'\bar{\partial}_b)S \sim S'S$$

and $S' \sim S'^*$ on W . We have on W

$$S' \sim SS' \sim (SS')^* = S'^*S^* \sim S'S \sim S.$$

Local existence

Let $S_1 = V^{-1}RR^*V$, $L_1 = V^{-1}L_0C^{-1}V$. Then we have

$$S_1^2 = V^{-1}RR^*RR^*V \sim V^{-1}RR^*V \sim S_1,$$

$$\bar{\partial}_b S_1 \sim V^{-1}CD_0V S_1 \sim V^{-1}CD_0VV^{-1}RR^*V \sim V^{-1}CD_0RR^*V \sim 0,$$

$$\begin{aligned} & S_1 + L_1\bar{\partial}_b \\ & \sim V^{-1}RR^*V + V^{-1}L_0C^{-1}VV^{-1}CD_0V \\ & \sim V^{-1}RR^*V + V^{-1}L_0D_0V \\ & \sim V^{-1}RR^*V + V^{-1}(I - RR^*)V \sim I. \end{aligned}$$

Since $R^*(V^{-1})^*V^{-1}R$ is an elliptic pseudo-differential operator, we denote by B its parametrix. Let $S = V^{-1}RBR^*(V^{-1})^*$. Then $B \sim B^*$ so $S \sim S^*$.

$$SS_1 = (V^{-1}RBR^*(V^{-1})^*)(V^{-1}RR^*V) = V^{-1}R(BR^*(V^{-1})^*V^{-1}R)R^*V \sim S_1$$

$$S_1 S = (V^{-1} R R^* V)(V^{-1} R B R^* (V^{-1})^*) \sim V^{-1} R B R^* (V^{-1})^* \sim S$$

$$\bar{\partial}_b S \sim \bar{\partial}_b S_1 S \sim 0$$

$S^2 \sim S S_1 S \sim S_1 S \sim S$. Let $L = (I - S)L_1$, then we have

$$S + L\bar{\partial}_b = S + (I - S)L_1\bar{\partial}_b \sim S + (I - S)(I - S_1) \sim S + I - S - S_1 + S S_1 \sim I.$$

Thus we show the existence of S .

Let $\{W_\alpha\}$ be an open covering of $T^*\tilde{U} - \{0\}$ and suppose we have S_α and L_α satisfying (1) and (2) on each W_α . By the same argument as in the proof of uniqueness we can show that on $W_\alpha \cap W_\beta$ we have $S_\alpha \sim S_\beta$. Using partition of unity as in section 2 we have Q_α with $\sum_\alpha Q_\alpha \sim I$ and $Q_\alpha \sim 0$ outside W_α .

Let $S = \sum Q_\alpha S_\alpha$ and $L = \sum Q_\alpha L_\alpha$. Then on W_β we have

$$S = \sum Q_\alpha S_\alpha \sim Q_\alpha S_\beta \sim S_\beta$$

which gives $S \sim S^* \sim S^2$. Also we have on each W_β

$$L\bar{\partial}_b = \sum Q_\alpha L_\alpha \bar{\partial}_b \sim \sum Q_\alpha (I - S_\alpha) \sim \sum Q_\alpha (I - S) \sim I - S.$$

This completes the proof of the proposition.

Such an S admits the following integral representation

$$Sf(x) = \int_{\tilde{U}} \int_0^\infty e^{it\psi(x,y)} a(x,y,t) f(y) dt dy,$$

where f has support in \tilde{U} and a is an symbol of degree n :

$$a(x,y,t) \sim \sum_{k=0}^\infty t^{n-k} a_k(x,y).$$

The kernel of S can be written as

$$S(x,y) = \int_0^\infty e^{it\psi(x,y)} a(x,y,t) dt,$$

and it is smooth off the diagonal. However S is only defined on \tilde{U} because $Sf(x)$ is not invariant under the action of the structure group G_U . Since we wish to have S defined on U instead of its local uniformization, we define

$$\tilde{S} = \frac{1}{|G_U|} \sum_{g \in G_U} g S g^{-1} \text{ i.e.}$$

$$\tilde{S}f(x) = \frac{1}{|G_U|} \sum_{g \in G_U} S\tilde{f}(gx) \text{ where } \tilde{f}(x) = f(g^{-1}x).$$

Now \tilde{S} admits the following integral representation

$$\begin{aligned}
\tilde{S}f(x) &= \sum_{g \in G_U} \int_{\tilde{U}} \int_0^\infty e^{it\psi(gx,y)} a(gx, y, t) f(g^{-1}y) dt dy \\
&= \sum_{g \in G_U} \int_{\tilde{U}} \int_0^\infty e^{it\psi(gx,gy)} a(gx, gy, t) f(y) dt dy \\
&= \sum_{g \in G_U} \int_{\tilde{U}} \int_0^\infty e^{it\psi(x,y)} a(gx, gy, t) f(y) dt dy.
\end{aligned}$$

The last equality holds since the volume form dy is G_U -invariant and ψ is invariant under the diagonal action of G_U . The kernel can be written as

$$\tilde{S}(x, y) = \sum_{g \in G_U} \int_0^\infty e^{it\psi(x,y)} a(gx, gy, t) dt = \int_0^\infty e^{it\psi(x,y)} \tilde{a}(x, y, t) dt$$

which is also smooth off the diagonal. We also define $\tilde{L} = \frac{1}{|G_U|} \sum gLg^{-1}$.

Now since \tilde{S} is G_U -equivariant, it can be considered as an operator defined on U instead of on \tilde{U} . The same is true for L .

Lemma 6.1 *On \tilde{U} we still have*

1. $\tilde{S} \sim \tilde{S}^* \sim \tilde{S}^2$
2. $\bar{\partial}_b \tilde{S} \sim 0$ and $I \sim \tilde{L} \bar{\partial}_b + \tilde{S}$,

where $A \sim B$ means $R = A - B$ is an operator of order $-\infty$ and is G_U -invariant (or equivalently its distribution kernel is C^∞ on $U \times U$).

Proof

$$\begin{aligned}
\bar{\partial}_b \tilde{S} &= \frac{1}{|G_U|} \sum_{g \in G_U} \bar{\partial}_b g S g^{-1} = \frac{1}{|G_U|} \sum g \bar{\partial}_b S g^{-1} \sim 0 \\
\tilde{L} \bar{\partial}_b &= \frac{1}{|G_U|} \sum g L \bar{\partial}_b g^{-1} \sim \frac{1}{|G_U|} \sum g (I - S) g^{-1} = I - \tilde{S} \\
\tilde{S}^* &= \frac{1}{|G|} \sum g S g^{-1} = \frac{1}{|G_U|} \sum (g S g^{-1})^* \\
&= \frac{1}{|G_U|} \sum (g^{-1})^* S^* g^* \sim \frac{1}{|G_U|} \sum g S g^{-1} \sim \tilde{S}
\end{aligned}$$

$$\tilde{S} \sim \tilde{L} \bar{\partial}_b \tilde{S} + \tilde{S}^2 \sim \tilde{S}^2.$$

Now let $\{U_j\}_{j \in J}$ be an open covering of X with their uniformization $\{U_j, G_j\}$ with partition of unity Q_j and let \tilde{S}_j and \tilde{L}_j be the corresponding operators on \tilde{U}_j . Let $S_X = \sum Q_j \tilde{S}_j$ and $L_X = \sum Q_j \tilde{L}_j$.

Definition If A and B are two operators defined on distribution on X , then by $A \sim B$ we mean $R = A - B$ is an operator of $-\infty$, or equivalently the kernel distribution of R is C^∞ on $X \times X$.

Lemma 6.2 *Given U_j and U_k , on the local uniformization $\tilde{U}_j \cap \tilde{U}_k$ the phase functions ψ_j and ψ_k of \tilde{S}_j and \tilde{S}_k are equivalent. Furthermore we have $\tilde{S}_j \sim \tilde{S}_k$.*

Notice that \tilde{S}_j and \tilde{S}_k have the same canonical relation on $T^*(\tilde{U}_j \cap \tilde{U}_k)$ so they can be composed with each other. And the claim can be easily shown by the same argument in the proof of the uniqueness in Proposition 6.4.

On each U_k we have $S_X = \sum Q_j \tilde{S}_j \sim \sum Q_j \tilde{S}_k \sim \sum Q_j \tilde{S}_k \sim \tilde{S}_k$ and hence

$$S_X^2 \sim S_X^* \sim S_X$$

$$\bar{\partial}_b S_X \sim 0,$$

and

$$L_X \bar{\partial}_b = \sum Q_j \tilde{L}_j \bar{\partial}_b \sim \sum Q_j (1 - \tilde{S}_j) \sim I - S_X.$$

So we have proved the following theorem:

Theorem 6.1 *There exist regular operators S_X and L_X on X such that*

1. $S_X \sim S_X^* \sim S_X^2$
2. $\bar{\partial}_b \sim 0$ and $I \sim L_X \bar{\partial}_b + S_X$,

and S_X is uniquely determined up to an operator of degree $-\infty$.

Let S denote the Szegö kernel of X . Then, by uniqueness of S_X and the fact that $I = S + Q_b \bar{\partial}_b$ we have $S \sim S_X$. For each point $(x_0, x_0) \in X \times X$ we can find a neighborhood U of x with its local uniformization $\{\tilde{U}, G_U\}$ such that the Szegö kernel S has the local representation

$$S_X(x, y) = \sum_{g \in G_U} \int_0^\infty e^{it\psi(gx, gy)} a(gx, gy, t) dt,$$

which is smooth off the diagonal. However $S_X(x, y)$ is not defined on $U \times U$. Remember for any distribution f defined on X , it is on a local uniformization $\{\tilde{U}, G_U\}$ G_U -invariant. So for f supported on U we have

$$\begin{aligned} S_X f(x) &= \sum_{g \in G_U} \int_{\tilde{U}} \int_0^\infty e^{it\psi(gx, gy)} a(gx, gy, t) f(y) dt dy \\ &= \sum_{g, h \in G_U} \int_{\tilde{U}} \int_0^\infty e^{it\psi(gx, gy)} a(gx, gy, t) \frac{1}{|G_U|} f(hy) dt dy \\ &= \sum_{g, h \in G_U} \int_{\tilde{U}} \int_0^\infty e^{it\psi(gx, hy)} a(gx, hy, t) \frac{1}{|G_U|} f(y) dt dy. \end{aligned}$$

This enables us to rewrite the Szegö kernel as

$$\Pi(x, y) = \frac{1}{|G_U|} \sum_{g, h \in G_U} \int_0^\infty e^{it\psi(gx, hy)} a(gx, hy, t) dt.$$

And such $\Pi(x, y)$ is well-defined on $U \times U$. Although the set of singularities of $\Pi(x, y)$ sit off-diagonal on $\tilde{U} \times \tilde{U}$, $\Pi(x, y)$ is smooth off the diagonal of $U \times U$.

Corollary 6.1 *For each point $(x_0, x_0) \in X \times X$ we can find a neighborhood U of x_0 with its local uniformization $\{\tilde{U}, G_U\}$ such that there exist smooth functions $F(x, y)$ and $G(x, y)$ on $\tilde{U} \times \tilde{U}$ such that the Szego kernel has the following representation*

$$S(x, y) = \sum_{g, h \in G_U} (F(gx, hy)(-i\psi(gx, hy))^{-n} + G(gx, hy) \log(-i\psi(gx, hy))).$$

7 Proof of the main theorems

On any uniformization (\tilde{U}, G) we choose a local holomorphic coframe e_L^* and let $a(z) = |e_L^*|_z^2$ and $(x, y) = (z, \lambda, w, \mu)$ on $X \times X$, we have $\rho(z, \lambda) = a(z)|\lambda|^2$ and

$$\psi(z, \lambda, w, \mu) = \frac{1}{i} a(z, w) \lambda \bar{\mu},$$

where $a(z, w)$ is an almost analytic function on $\tilde{U} \times \tilde{U}$ satisfying $a(z, z) = a(z)$.

On X we have $a(z)|\lambda|^2 = 1$, so we can assume that $\lambda = a(z)^{-\frac{1}{2}} e^\theta$, then

$$\psi(z, \theta_1, w, \theta_2) = \frac{1}{i} \left(\frac{a(z, w)}{i\sqrt{a(z)}\sqrt{a(w)}} e^{i(\theta_1 - \theta_2)} - 1 \right).$$

The weight space projections Π_N are Fourier coefficients of Π and hence can be written as

$$\begin{aligned}\Pi_N(x, y) &= \sum_{g, h \in G} \int_{t=0}^{\infty} \int_{S^1} e^{-iN\theta - it\psi(r_\theta gx, hy)} s(r_\theta gx, hy, t) d\theta dt \\ &= \sum_{g, h \in G} \int_{t=0}^{\infty} \int_{S^1} e^{iN(-\theta + t\psi(r_\theta gx, hy))} s(r_\theta gx, hy, Nt) d\theta dt.\end{aligned}$$

In particular on the diagonal $x = y$, we have

$$\psi(r_\theta gx, hx) = \frac{1}{i} \left(\frac{a(gz, hz)}{a(z)} e^{i\theta} - 1 \right).$$

So the phase $\psi(t, \theta; gx, hx) = \frac{t}{i} \left(\frac{a(gz, hz)}{a(z)} e^{i\theta} - 1 \right) - \theta$. If $g = h$, then $\Psi(t, \theta; x, x) = \frac{t}{i} (e^{i\theta} - 1) - \theta$. We have $d_t \Psi(t, \theta; x, x) = \frac{1}{i} (e^{i\theta} - 1)$ and $d_\theta \Psi(t, \theta; x, x) = te^{i\theta} - 1$ thus the critical set is $\{\theta = 0, t = 1\}$. The Hessian Ψ'' on the critical set is equal to $\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$. So the phase is nondegenerate and the critical points are independent of x and we can apply the theorem in [14]. However if $g \neq h$ there is no critical point except $z = 0$ and we cannot apply the theorem and this makes the asymptotic expansion fail near the singularities.

$$\begin{aligned}\Pi_N(z) &= \sum_{g, h \in G} \int_{t=0}^{\infty} \int_{S^1} e^{iN(-\theta + t\psi(r_\theta gx, hx))} s(r_\theta gx, hx, Nt) d\theta dt \\ &= \sum_{g=h \in G} \int_{t=0}^{\infty} \int_{S^1} e^{-iN\theta - it\psi(r_\theta gx, hx)} s(r_\theta gx, hx) d\theta dt \\ &\quad + \sum_{g \neq h \in G} \int_{S^1} e^{-iN\theta} \frac{F_{g, h, \theta}}{\left(1 - \frac{a(gz, hz)}{a(z)} e^{i\theta}\right)^{n+1}} d\theta \\ &\quad + \sum_{g \neq h \in G} \int_{S^1} e^{-iN\theta} E_{g, h, \theta} \log\left(1 - \frac{a(gz, hz)}{a(z)} e^{i\theta}\right) d\theta \\ &= \Pi_N^{(1)}(z) + \Pi_N^{(2)}(z) + \Pi_N^{(3)}(z).\end{aligned}$$

$\Pi_N^{(1)}$ has a converging expansion similar in [32] while $\Pi_N^{(2)}$ might cause difficulties near the singular points.

$$\begin{aligned}\Pi_N^{(2)}(z) &= \sum_{g \neq h \in G} \int_{S^1} e^{-iN\theta} \frac{F_{g, h, \theta}}{\left(1 - \frac{a(gz, hz)}{a(z)} e^{i\theta}\right)^{n+1}} d\theta \\ &= \sum_{g \neq h \in G} \int_{S^1} e^{-iN\theta} F_{g, h, \theta} \left(\sum_{k=0}^{\infty} \frac{(k+n)!}{n!k!} \left(\frac{a(gz, hz)}{a(z)} \right)^k e^{ik\theta} \right) d\theta \\ &= \sum_{g \neq h \in G} \sum_{k=-\infty}^N \frac{(l+n)!}{n!l!} f_{g, h}^{(k)} \left(\frac{a(gz, hz)}{a(z)} \right)^{N-k},\end{aligned}$$

and

$$\begin{aligned}
\Pi_N^{(3)}(z) &= \sum_{g \neq h \in G} \int_{S^1} e^{-iN\theta} E_{g,h,\theta} \log\left(1 - \frac{a(gz, hz)}{a(z)} e^{i\theta}\right) d\theta \\
&= \sum_{g \neq h \in G} \int_{S^1} e^{-iN\theta} E_{g,h,\theta} \left(\sum_{k=0}^{\infty} \frac{1}{k} \left(\frac{a(gz, hz)}{a(z)}\right)^k e^{ik\theta}\right) d\theta \\
&= \sum_{g \neq h \in G} \sum_{k+l=N} \frac{1}{l} e_{g,h}^{(k)} \left(\frac{a(gz, hz)}{a(z)}\right)^l
\end{aligned}$$

Here we assume $F_{g,h,\theta} = \sum_k f_{g,h}^{(k)} e^{ik\theta}$ and $E_{g,h,\theta} = \sum_k e_{g,h}^{(k)} e^{ik\theta}$.

Lemma 7.1 *There exists $\delta > 0$ such that for any positive integers l and s there is a constant $C_{l,s}$ such that*

1. $\|\Pi_N^{(2)}\|_{C^s} \leq C_{s,l} \left(\frac{1}{N^l} + N^{n+s/2} e^{-\delta N r^2}\right)$.
2. $\|\Pi_N^{(3)}\|_{C^s} \leq C_{s,l} \left(\frac{1}{N^l} + N^{s/2-1} e^{-\delta N r^2}\right)$.

Proof

$$\begin{aligned}
|\Pi_N^{(2)}| &\leq \sum_{g \neq h \in G} \sum_{k=-\infty}^N \frac{(N-k+n)!}{n!(N-k)!} |f_{g,h}^{(k)}| \left|\frac{a(gz, hz)}{a(z)}\right|^{N-k} \\
&\leq \sum_{g \neq h \in G} \sum_{|k| \geq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} |f_{g,h}^{(k)}| \left|\frac{a(gz, hz)}{a(z)}\right|^{N-k} \\
&\quad + \sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} |f_{g,h}^{(k)}| \left|\frac{a(gz, hz)}{a(z)}\right|^{N-k} \\
&\leq \sum_{g \neq h \in G} \left\{ C_1 \sum_{|k| \geq \sqrt{N}} (N-k+n)^n |f_{g,h}^{(k)}| + C_2 \sum_{|k| \leq \sqrt{N}} N^n |f_{g,h}^{(k)}| \left|\frac{a(gz, hz)}{a(z)}\right|^{N-k} \right\}.
\end{aligned}$$

Since $|f_{g,h}^{(k)}| \leq C_l k^l$, we have

$$\begin{aligned}
|\Pi_N^{(2)}| &\leq C_l \frac{1}{N^l} + C_2 N^n \left(\sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} |f_{g,h}^{(k)}| e^{(N-k) \log \left|\frac{a(gz, hz)}{a(z)}\right|} \right) \\
&\leq C_l \frac{1}{N^l} + C_3 N^n \left(\sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} |f_{g,h}^{(k)}| \right) e^{-\delta N r^2} \\
&\leq C_{l,0} \left(\frac{1}{N^l} + N^n e^{-\delta N r^2} \right),
\end{aligned}$$

and

$$|\Pi_N^{(3)}| \leq \sum_{g \neq h \in G} \sum_{k=-\infty}^N \frac{1}{N-k} |e_{g,h}^{(k)}| \left|\frac{a(gz, hz)}{a(z)}\right|^{N-k}$$

$$\begin{aligned}
&\leq \sum_{g \neq h \in G} \left\{ \sum_{|k| \geq \sqrt{N}} \frac{1}{N-k} |e_{g,h}^{(k)}| + C_4 \sum_{|k| \leq \sqrt{N}} \frac{1}{N} |f_{g,h}^{(k)}| e^{(N-k) \log \left| \frac{a(gz, hz)}{a(z)} \right|} \right\} \\
&\leq C_l \frac{1}{N^l} + C_4 \frac{1}{N} \sum_{g \neq h \in G} e^{(N-k) \log \left| \frac{a(gz, hz)}{a(z)} \right|} \\
&\leq C_l \frac{1}{N^l} + C_4 \frac{1}{N} \left(\sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} |f_{g,h}^{(k)}| \right) e^{-\delta N r^2} \\
&\leq C_{l,0} \left(\frac{1}{N^l} + \frac{1}{N} e^{-\delta N r^2} \right).
\end{aligned}$$

We have similar bounds for $|\Pi_N^{(2)}|_{C^k}$ and $|\Pi_N^{(3)}|_{C^k}$. If $N > \frac{1}{r^2}$ where r is the distance from x to the singular set. Hence $\Pi_N^{(2)}$ and $\Pi_N^{(3)}$ converges to 0 uniformly away from the singular point. This also proves the first part of Theorem 1.2.

Lemma 7.2 *Suppose z_0 is an isolated singularity on X . Then there is a constant $b(z_0)$ such that for any C^∞ test function $\phi(z)$ supported in a small neighborhood of z_0 , with z_0 its only singularity, we have*

$$\lim_{N \rightarrow \infty} \langle \Pi_N^{(2)}(z) + \Pi_N^{(3)}(z), \phi(z) \rangle = \lim_{N \rightarrow \infty} \int_M (\Pi_N^{(2)}(z) + \Pi_N^{(3)}(z)) \phi(z) \omega^n = b(z_0) \phi(z_0),$$

where $b(z_0) = \sum_{1 \neq g \in G_{z_0}} \frac{1}{\det(I-gT_{z_0})}$.

Proof

$$\begin{aligned}
&\int_{|z| \leq \epsilon} \Pi_N^{(2)} \phi dV \\
&= \sum_{g \neq h \in G} \int_{|z| \leq \epsilon} \sum_{|k| \leq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} f_{g,h}^{(k)} \left(\frac{1+gz\bar{h}z}{1+|z|^2} \right)^{N-k} \phi dV \\
&= \sum_{g \neq h \in G} \int_{|z| \leq \epsilon} \sum_{|k| \leq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} f_{g,h}^{(k)} \left(\frac{1+gz\bar{h}z}{1+|z|^2} \right)^{N-k} \phi(0) dV \\
&\quad + O\left(\left| \sum_{g \neq h \in G} \int_{|z| \leq \epsilon} \sum_{|k| \leq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} f_{g,h}^{(k)} \left(\frac{1+gz\bar{h}z}{1+|z|^2} \right)^{N-k} |z| dV \right| \right) \\
&= \left\{ \sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} \int_{r=0}^{\epsilon} \int_{S^{2n-1}} \frac{(N-k+n)!}{n!(N-k)!} \left(1 - \frac{r(1-\xi g \bar{h} \bar{\xi})}{1+r} \right)^{N-k} d\xi r^{n-1} dr f_{g,h}^{(k)}(0) \right\} \phi(0) \\
&\quad + O\left(\left| \sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} \int_{r=0}^{\epsilon} \int_{S^{2n-1}} \frac{(N-k+n)!}{n!(N-k)!} \left| 1 - \frac{r(1-\xi g \bar{h} \bar{\xi})}{1+r} \right|^{N-k} d\xi r^n dr \right| \right)
\end{aligned}$$

Let $s = \frac{r}{1+r}$. We have

$$\begin{aligned}
& \int_{|z| \leq \epsilon} \Pi_N^{(2)} \phi dV \\
&= \left\{ \int_{S^{2n-1}} \sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} \int_{s=0}^{\frac{\epsilon}{1+\epsilon}} \frac{(N-k+n)!}{n!(N-k)!} (1 - (1 - \xi g \bar{h} \bar{\xi})s)^{N-k} s^{2n-1} ds f_{g,h}^{(k)}(0) d\xi \right\} \phi(0) \\
&\quad + O\left(\frac{1}{N}\right) \\
&= \left\{ \int_{S^{2n-1}} \sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} \frac{n!}{(N-k)\dots(N-k+n-1)} (1 - \xi g \bar{h} \bar{\xi})^{-(n-1)} \right. \\
&\quad \cdot \left. \int_{s=0}^{\frac{\epsilon}{1+\epsilon}} (1 - (1 - \xi g \bar{h} \bar{\xi})s)^{N-k+n-1} ds f_{g,h}^{(k)}(0) d\xi \right\} \phi(0) + O\left(\frac{1}{N}\right) \\
&= \left\{ \int_{S^{2n-1}} \sum_{g \neq h \in G} \sum_{|k| \leq \sqrt{N}} (1 - \xi g \bar{h} \bar{\xi})^n f_{g,h}^{(k)}(0) d\xi \right\} \phi(0) + O\left(\frac{1}{N}\right) \\
&= \left\{ \sum_{g \neq h \in G} \left(\int_{S^{2n-1}} \frac{1}{(1 - \xi g \bar{h} \bar{\xi})^n} d\xi \right) \left(\sum_{|k| \leq \sqrt{N}} f_{g,h}^{(k)}(0) \right) \right\} \phi(0) + O\left(\frac{1}{N}\right).
\end{aligned}$$

Taking $N \rightarrow \infty$, we have

$$\begin{aligned}
& \int_{|z| \leq \epsilon} \Pi_N^{(2)} \phi dV \\
&\rightarrow \left(\sum_{g \neq h \in G} \int_{S^{2n-1}} \frac{1}{(1 - \xi g \bar{h} \bar{\xi})^n} d\xi \right) F((z_0, 0), (z_0, 0)) \phi(z_0) \\
&= b(z_0) \phi(z_0).
\end{aligned}$$

The last equation comes from the fact that $F((z_0, 0), (z_0, 0)) = 1$. Also we have $\int_{|z| \leq \epsilon} \Pi_N^{(3)} \phi dV = O\left(\frac{1}{N}\right)$ which converges to 0 as $N \rightarrow \infty$.

We know that $\Sigma_N^{(1)}$ has an asymptotic expansion: $\|\Sigma_N^{(1)} - (a_0 N^n + a_1(z) N^{n-1} + \dots + a_n(z))\|_{C^k} \leq C_k N^{-1}$. So $\Pi_N(z) - (a_0 N^n + a_1(z) N^{n-1} + \dots + a_n(z))$ converges to $\sum_{i=1}^m b(i) \delta_{z_i}(z)$. Therefore we prove Theorem 1.2.

Corollary 7.2 $\chi_N = \frac{1}{N} \log(\Pi_N)$ converges to 0 in $C^0(M)$.

Proof It suffices to show that Π_N is bounded from below by a uniform positive constant for N large. This can be shown by constructing equivariant peak sections as in [26, 27].

Actually we can obtain much stronger result for the special case where the structure groups are all abelian and from now on we assume X only has finite isolated singularities with abelian structure groups.

Lemma 7.3 *If G is a finite abelian subgroup of $U(n)$ then there exists a uniform constant $C_{G,n} > 0$ such that for integer $N > 0$ and any $z \in C^n$ we have*

$$\sum_{g \in G} \left(\frac{1 + gz\bar{z}}{1 + |z|^2} \right)^N > C_{G,n}.$$

Proof Since G is a finite abelian group of isometry of T_{z_0} , G can be linearized as a finite abelian subgroup of $U(n)$ and therefore all elements of G can be diagonalized at the same time. Any element $g \in G$ can be expressed as

$$g = \begin{pmatrix} e^{\frac{i2p_1(g)\pi}{q_1(g)}} & 0 & \dots & 0 \\ 0 & e^{\frac{i2p_2(g)\pi}{q_2(g)}} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{\frac{i2p_n(g)\pi}{q_n(g)}} \end{pmatrix},$$

where p_i and q_i are relatively prime for $i = 1, \dots, n$.

Then

$$\begin{aligned} & \sum_{g \in G} (1 + gz\bar{z})^N \\ &= \sum_{g \in G} \sum_{\alpha_0 + \alpha_1 + \dots + \alpha_n = N} \frac{N!}{\alpha_0! \alpha_1! \dots \alpha_n!} |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} \dots |z_n|^{2\alpha_n} e^{\frac{i2\alpha_1 p_1(g)\pi}{q_1(g)} + \frac{i2\alpha_2 p_2(g)\pi}{q_2(g)} + \frac{i2\alpha_n p_n(g)\pi}{q_n(g)}} \\ &= \sum_{\alpha_0 + \alpha_1 + \dots + \alpha_n = N} \frac{N!}{\alpha_0! \alpha_1! \dots \alpha_n!} |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} \dots |z_n|^{2\alpha_n} \left[\sum_{g \in G} e^{\frac{i2\alpha_1 p_1(g)\pi}{q_1(g)} + \frac{i2\alpha_2 p_2(g)\pi}{q_2(g)} + \frac{i2\alpha_n p_n(g)\pi}{q_n(g)}} \right] \end{aligned}$$

Fix $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$, we can construct the following group homomorphism

$$\begin{aligned} \alpha : G &\rightarrow U(1) \\ \alpha(g) &= e^{\frac{i2\alpha_1 p_1(g)\pi}{q_1(g)} + \frac{i2\alpha_2 p_2(g)\pi}{q_2(g)} + \frac{i2\alpha_n p_n(g)\pi}{q_n(g)}}. \end{aligned}$$

It is easy to see that $\sum_{g \in G} e^{\frac{i2\alpha_1 p_1(g)\pi}{q_1(g)} + \frac{i2\alpha_2 p_2(g)\pi}{q_2(g)} + \frac{i2\alpha_n p_n(g)\pi}{q_n(g)}}$ is nonzero only if α is a trivial homomorphism and in this case $\sum_{g \in G} e^{\frac{i2\alpha_1 p_1(g)\pi}{q_1(g)} + \frac{i2\alpha_2 p_2(g)\pi}{q_2(g)} + \frac{i2\alpha_n p_n(g)\pi}{q_n(g)}}$ is a positive integer. Therefore

$$\begin{aligned} & \sum_{g \in G} (1 + gz\bar{z})^N \\ & \geq \sum_{g \in G} \sum_{\substack{\alpha_0 + \alpha_1 + \dots + \alpha_n = N \\ |G| \alpha_k, \text{ for } k = 1, \dots, n}} \frac{N!}{\alpha_0! \alpha_1! \dots \alpha_n!} |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} \dots |z_n|^{2\alpha_n} \\ & \geq C_{G,n} (1 + |z|^2)^N. \end{aligned}$$

Lemma 7.4 *There exist constants N_0, C and $c > 0$ such that for $N > N_0$ we have for all $z \in M$*

$$cN^n \leq \Pi_N(z) \leq CN^n.$$

Proof For the proof of Theorem 1.3, we have the upper bound. For the lower bound it suffices to prove it near the singularities.

$$\begin{aligned} \Pi_N &= \sum_{g,h \in G} \sum_{k=-\infty}^N \frac{(N-k+n)!}{n!(N-k)!} f_{g,h}^{(k)} \left(\frac{a(gz, hz)}{a(z)} \right)^{N-k} \\ &= \sum_{g,h \in G} \sum_{|k| \leq \sqrt{N}} \frac{(N-k+n)!}{n!(N-k)!} f_{g,h}^{(k)} \left(\frac{a(gz, hz)}{a(z)} \right)^{N-k} + O(N^{n-1}) \\ &= \sum_{g,h \in G} \sum_{|k| \leq \sqrt{N}} \frac{N^n}{n!} f_{g,h}^{(k)} \left(\frac{a(gz, hz)}{a(z)} \right)^{N-k} + O(N^{n-1}) \\ &= \sum_{g,h \in G} \sum_{|k| \leq \sqrt{N}} \frac{N^n}{n!} f_{g,h}^{(k)}(z=0) \left(\frac{a(gz, hz)}{a(z)} \right)^{N-\sqrt{N}} + O(N^{n-\epsilon'}) \\ &= \frac{N^n}{n!} \sum_{|k| \leq \sqrt{N}} f^{(k)}(z=0) \sum_{g \neq h \in G} \left(\frac{a(gz, hz)}{a(z)} \right)^{N-\sqrt{N}} + O(N^{n-\epsilon'}) \\ &\geq C_5 N^n \sum_{g \in G} \left(\frac{1+gz\bar{z}}{1+|z|^2} \right)^{N-\sqrt{N}} + O(N^{n-\epsilon'}) \geq cN^n \end{aligned}$$

by lemma 7.3 and here $\sum_{|k| \leq \sqrt{N}} f^{(k)}(z=0)$ converges to $F(0,0)$ which is positive.

Proposition 7.1 $\chi_N = \frac{1}{N} \log \Pi_N$ converges to 0 in $C^{1,\alpha}(M)$ for any $\alpha < 1$.

Proof

$$\nabla \chi_N = \frac{1}{N} \frac{\nabla \Pi_N^{(1)} + \nabla \Pi_N^{(2)} + \nabla \Pi_N^{(3)}}{\Pi_N}.$$

Since we have the above lemma we can do the same calculation in Lemma 7.1 and it is straightforward to prove the lemma.

Proposition 7.2 *Let $\omega(N) = \frac{1}{N} \log(\sum_{i=0}^{d_N} \|S_I^N(z)\|_{h_N}^2)$ be the pullback of the scaled Fubini-Study metric then*

$$\|\omega(N) - \omega\|_{C^k} \leq C_k \left(\frac{1}{N} + \left(\sum_{l=0}^k N^{k/2} (Nr^2)^{l/2+1} \right) e^{-\delta Nr^2} \right).$$

In particular, $\|\omega(N) - \omega\|_{C^k} \leq C'_k N^{k/2}$.

The proof is straightforward by induction.

Let $P(M, \omega) = \{\phi \in C^\infty(M) \mid \omega_\phi = \omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0, \sup_M \phi = 0\}$ be the set of all plurisubharmonic functions on M . Let

$$\begin{aligned}\tilde{\omega}_\phi &= \omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0 \\ \tilde{h} &= h e^{-\phi}.\end{aligned}$$

Let \tilde{h}_N be the induced Hermitian metric of \tilde{h} on L^N , $\{\tilde{S}_0^m, \tilde{S}_1^N, \dots, \tilde{S}_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^N)$ ($1 + d_N = \dim H^0(M, L^N)$) with respect to \tilde{h}, ω_ϕ . Then we have the following holomorphic approximation theorem which has been proved by Lu.

Corollary 7.1 $\phi_N = \frac{1}{N} \log(\sum_{k=0}^{d_N} \|\tilde{S}_k^N(z)\|_{\tilde{h}_N}^2)$ converges to ϕ in $C^0(M)$ for any $\alpha < 1$. If for each singularity its structure group is cyclic, then the ϕ_N converges to ϕ in $C^{1,\alpha}(M)$ for any $\alpha < 1$.

8 Examples

Let X be the quotient of \mathbf{CP}^1 by a cyclic group of order n defined by

$$[Z_0, Z_1] \sim [Z_0 e^{\frac{i2k\pi}{n}}, Z_1]$$

for $k = 0, 1, 2, \dots, n-1$.

Let L be $O(n)$ the orbifold line bundle over X and the Fubini-Study metric on X is defined by $g_{i\bar{j}} = \frac{\sqrt{-1}}{2\pi} \partial_i \bar{\partial}_{\bar{j}} \log(|Z_0|^2 + |Z_1|^2)$. Then $\{\sqrt{\frac{n(nN+1)!}{(nk)!(nN-nk)!}} Z_0^{nk} Z_1^{nN-nk}\}_{k=0}^N$ is an orthonormal basis of $H^0(X, O(nN))$ with respect to the Fubini-Study metric.

On the patch $U_0 = \{Z_0 \neq 0\}$ we write $r = \frac{|Z_1|^2}{|Z_0|^2}$ and we have

$$\begin{aligned}S^{(N)}(z) &= n \sum_{m=0}^N (nN+1) C_{nN}^{mm} \frac{|Z_0|^{2nm} |Z_1|^{2nN-2nm}}{(|Z_0|^2 + |Z_1|^2)^{nN}} \\ &= n \sum_{k=m}^N (nN+1) C_{nN}^{mm} \frac{r^{nm}}{(1+r)^{nN}} \\ &= (nN+1) \sum_{k=0}^{n-1} \left(\frac{1 + r e^{\frac{i2k\pi}{n}}}{1+r} \right)^{nN} \\ &= nN + 1 + nN \sum_{k=1}^{n-1} \left(\frac{1 + r e^{\frac{i2k\pi}{n}}}{1+r} \right)^{nN}.\end{aligned}$$

And

$$|S^{(N)}|([1, 0]) = |S^{(N)}|^2([0, 1]) = n(nN + 1)$$

$$\lim_{N \rightarrow \infty, [Z_0, Z_1] \neq [0, 1], [1, 0]} \frac{|S^{(N)}|^2([Z_0, Z_1])}{nN + 1} = 1.$$

Also we have

$$\begin{aligned} |S^{(N)}(z) - (nN + 1)| &= (nN + 1) \left| \sum_{k=1}^{n-1} \left(\frac{1 + re^{\frac{i2k\pi}{n}}}{1 + r} \right)^{nN} \right| \\ &\leq n(nN + 1) \max \left| 1 - \frac{r(1 - e^{\frac{i2k\pi}{n}})}{1 + r} \right|^{nN} \\ &\leq n(nN + 1)e^{-\delta Nr} \\ &\leq C_2 \frac{1}{N}, \end{aligned}$$

if $N \geq \frac{1}{r^2}$.

By similar calculation we have

$$S^{(N)} \sim nN + 1 + \frac{n-1}{2n} (\{[0, 1]\} + \{[1, 0]\}),$$

as a distribution on X and if we integrate $S^{(N)}$ over X we obtain the Gauss-Bonnet theorem for the quotient sphere. It also verifies Theorem 1.2 since $a(0) = 1$, $a(1) = 1$ as the scalar curvature and

$$b_{[0,1]} = b_{[1,0]} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{1 - e^{\frac{i2k\pi}{n}}} = \frac{n-1}{2n}.$$

References

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