ON THE CONVERGENCE AND SINGULARITIES
OF THE J-FLOW WITH APPLICATIONS
TO THE MABUCHI ENERGY

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1. Introduction

The J-flow is a parabolic flow on Kähler manifolds with two Kähler classes. It was discovered by Donaldson [Do1] in the setting of moment maps and by Chen [Ch2] as the gradient flow of the $J$-functional appearing in his formula for the Mabuchi energy [Ma1].

The J-flow is defined as follows. Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$ and let $\chi_0$ be another Kähler form on $M$. Let $\mathcal{H}$ be the space of Kähler potentials

$$\mathcal{H} = \{ \phi \in C^\infty(M) \mid \chi_\phi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi > 0 \}.$$ 

The J-flow is the flow in $\mathcal{H}$ given by

$$\frac{\partial \phi}{\partial t} = c - \omega \wedge \chi_\phi^{n-1} \chi_\phi^n,$$

$$\phi|_{t=0} = \phi_0 \in \mathcal{H},$$

where $c$ is the constant given by

$$c = \frac{[\omega] \cdot [\chi_0]^{n-1}}{[\chi_0]^n}.$$ 

A critical point of the J-flow gives a Kähler form $\chi$ satisfying

$$\omega \wedge \chi^{n-1} = c \chi^n.$$ (1.2)

In local coordinates, this critical equation can be written

$$\chi^{\bar{j}} g_{\bar{j}} = nc,$$
where $\chi_{ij}$ and $g_{ij}$ are the Kähler metrics corresponding to $\chi$ and $\omega$. This is a fully nonlinear second order elliptic equation in $\phi$. If a solution $\chi > 0$ of (1.2) exists then it is the unique solution in its class [Ch2, Do1].

Choosing a normal coordinate system for $g$ so that $\chi_{ij}$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$, the critical equation becomes

$$\sum_{i=1}^{n} \frac{1}{n c \lambda_i} = 1.$$  

Donaldson [Do1] noted that the necessary condition

$$\frac{1}{n c \lambda_i} < 1 \quad \text{for } i = 1, \ldots, n,$$  

translates into the Kähler class condition

$$[n c \chi_0 - \omega] > 0.$$  

He also remarked that an obvious conjecture would be that this be sufficient for the existence of a critical metric (see also [Ch2], Conjecture/Question 1). Chen [Ch2] confirmed this conjecture in the case $n = 2$ by reducing (1.2) to the complex Monge-Ampère equation which was solved by Yau [Ya1].

For a general Kähler manifold, Chen [Ch3] showed that solutions of the J-flow exist for all time for any smooth initial data, and proved convergence in the case of non-negative bisectional curvature. The second author showed in [We1, We2] that the J-flow converges to a critical metric under the condition

$$n c \chi_0 - (n - 1) \omega > 0,$$  

which in coordinates as above corresponds to

$$\frac{1}{n c \lambda_i} < \frac{1}{n - 1} \quad \text{for } i = 1, \ldots, n.$$  

This shows that the critical equation can be solved under the class condition

$$[n c \chi_0 - (n - 1) \omega] > 0,$$  

which coincides with the above conjecture for $n = 2$.

In this paper we find a necessary and sufficient condition for the convergence of the flow and existence of a critical metric. In terms of the $\lambda_i$, this condition can be written

$$\sum_{i=1}^{n} \frac{1}{n c \lambda_i} < 1 \quad \text{for } k = 1, \ldots, n.$$  

\[2\]
Clearly (1.4) implies (1.5) implies (1.3), and all three coincide for the case $n = 2$. This condition can be rewritten in terms of the positivity of a certain $(n - 1, n - 1)$-form. Precisely, we prove the following.

**Theorem 1.1** Let $M$ be a compact Kähler manifold of complex dimension $n$ with two Kähler metrics $\omega$ and $\chi_0$. The following are equivalent:

(i) There exists a metric $\chi'$ in $[\chi_0]$ satisfying

$$(nc\chi' - (n - 1)\omega) \wedge \chi'^{n-2} > 0.$$  \hspace{1cm} (1.6)

(ii) For any initial data $\phi_0$ in $\mathcal{H}$, the $J$-flow $(1.1)$ converges in $C^\infty$ to $\phi_\infty$ in $\mathcal{H}$ with $\chi = \chi_{\phi_\infty}$ satisfying the critical equation (1.2).

(iii) There exists a smooth solution $\chi$ in $[\chi_0]$ to the critical equation (1.2).

Recall that an $(n-1, n-1)$ form $v$ is positive if for all $(1, 0)$ forms $\alpha, \sqrt{-1}v \wedge \alpha \wedge \overline{\alpha} > 0$.

Notice that the condition (1.6) is satisfied if the class condition $[nc\chi_0 - (n - 1)\omega] > 0$ holds.

In [Ch2], Chen showed using results from [Ch1] that finding a solution of the critical equation implies the lower boundedness of the Mabuchi energy [Ma1] on certain Kähler classes for manifolds with negative first Chern class. We will now briefly discuss this functional and its role in Kähler geometry.

The Mabuchi energy is the functional on $\mathcal{H}$ given by

$$M_{\chi_0}(\phi) = - \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} (R_{\chi_{\phi_t}} - \overline{R}) \frac{\chi_{\phi_t}^n}{n!} dt,$$

where $\{\phi_t\}_{0 \leq t \leq 1}$ is a path in $\mathcal{H}$ between $0$ and $\phi$, $R_{\chi_{\phi_t}}$ is the scalar curvature of $\chi_{\phi_t}$ and $\overline{R}$ is the average of the scalar curvature, given by

$$\overline{R} = \frac{1}{\int_M \chi_0^n} \int_M R_{\chi_0} \chi_0^n.$$

The critical points of the Mabuchi energy are constant scalar curvature Kähler (cscK) metrics. The question of the existence of a cscK metric in a given Kähler class is a difficult and interesting problem and is expected
to be equivalent to a notion of stability in the sense of geometric invariant theory. This is an idea of Yau [Ya2], who made the conjecture for Kähler-Einstein metrics on Fano manifolds. In recent years there has been much progress on this problem, in particular by Tian [Ti3] and Donaldson [Do2]. There is now a large body of literature pertaining to Yau’s conjecture and we refer the reader to the additional references [Ti2, Ti4, DiTi, Do3, Do4, Lu, PhSt1, PhSt2, PhSt3, PaTi, Ma2, RoTh], which is far from a complete list, for details. The Mabuchi energy and its ‘derivative’, the Futaki invariant [Fu], are central to these ideas.

It is known that if there exists a Kähler-Einstein metric in a class [\(\chi_0\)] then the Mabuchi energy is bounded below on that class. This was shown by Bando and Mabuchi [BaMa] for the case \(c_1(M) > 0\). The proof for the cases \(c_1(M) < 0\) and \(c_1(M) = 0\) can be found in [Ti4]. Moreover, Chen and Tian [ChTi] have recently shown that the existence of a cscK metric in any class implies the lower boundedness of Mabuchi’s functional. In particular, a manifold with \(c_1(M) < 0\) admits a Kähler-Einstein metric [Ya1, Au1] and so the Mabuchi energy is bounded below on any Kähler class which is a positive multiple of \(-c_1(M)\). For manifolds with \(c_1(M) = 0\), there exists a Ricci-flat metric [Ya1] in any class and so the Mabuchi energy is bounded below on any \([\chi_0]\).

More generally, Tian [Ti4] has conjectured that the existence of a cscK metric is equivalent to the ‘properness’ of the Mabuchi energy. This means that it is bounded below by a certain energy functional. For the precise definition, see section 2. This result is already known [Ti3, Ti4, TiZh] when \(c_1(M)\) is a multiple of \([\chi_0]\).

We will deal with the case when \(M\) has negative first Chern class and when \([\chi_0]\) is not necessarily a multiple of \(-c_1(M)\). It was shown in [We2], using the J-flow, that if \(c_1(M) < 0\) then the Mabuchi energy is bounded below on all Kähler classes \([\chi_0]\) satisfying

\[-n c_1(M) \cdot [\chi_0]^{n-1} [\chi_0] + (n - 1) c_1(M) > 0.\]

Chen [Ch2] had proved the case \(n = 2\). Since this condition is easily satisfied for \([\chi_0] = -c_1(M)\) it follows that this inequality defines a reasonably large open conical neighbourhood of \(-c_1(M)\) in the Kähler cone. Using Theorem 1.1 we can enlarge the set of classes for which this holds and, using a result of Tian [Ti4], show that the Mabuchi energy is not just bounded below but is in fact proper.
Theorem 1.2 Suppose that $M$ satisfies $c_1(M) < 0$. Let $V$ be the cone of all Kähler classes $[\chi_0]$ with the property that there exist metrics $\omega$ in $-\pi c_1(M)$ and $\chi'$ in $[\chi_0]$ with

$$
\left(-n \frac{\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} \chi' - (n-1) \omega\right) \wedge \chi'^{n-2} > 0.
$$

Then the Mabuchi energy is proper on every class $[\chi_0]$ in $V$.

In section 2 we explain the relationship between the Mabuchi energy and the J-functional which leads to this result.

Returning to the J-flow, we also consider the case when the inequality (1.6) is not satisfied. Using a method similar to that used by Tsuji [Ts] for the Kähler-Ricci flow, we can still obtain some estimates away from a subvariety if we make more general assumptions. Moreover, under these assumptions, and if (1.6) is not satisfied, we show that the flow must blow up over a subvariety in a certain sense.

Theorem 1.3 Let $M$ be a compact Kähler manifold of complex dimension $n$ with two Kähler metrics $\omega$ and $\chi_0$. Suppose that there exists an effective real divisor $D = \sum_{\nu=1}^m a_\nu S_\nu$ on $M$ and a metric $\chi'$ in the class $([\chi_0] - (1/nc)[D])$ satisfying

$$(nc\chi' - (n-1)\omega) \wedge \chi'^{n-2} > 0.$$ 

Let $\phi_t$ be a solution of the J-flow (1.1) and let $S = \cup_{\nu=1}^m S_\nu$. For any holomorphic sections $s_\nu$ of the line bundles associated to $S_\nu$, vanishing on $S_\nu$, and for any hermitian metrics $h_\nu$ on these line bundles, there exist constants $C, C'$ and $A$ depending only on $D, \omega, \chi_0, \phi_0, s_\nu$ and $h_\nu$ such that

(a) $\phi_t(x) \geq -C + \sum_{\nu=1}^m \frac{a_\nu}{nc\pi} \log |s_\nu|^2_{h_\nu}(x), \quad \text{for } x \in M - S$;

(b) $\Lambda_\omega \chi_{\phi_t}(x) \leq \frac{C'}{|s_1|^{2A_1/nc\pi} \cdots |s_m|^{2A_m/nc\pi}(x)} e^{A \phi_t(x)}, \quad \text{for } x \in M - S$.

Let $\tilde{S}$ be the intersection of all sets $S$ corresponding to divisors in the linear system $[D]$. Then, with the same assumptions as above, if there does not exist $\chi'$ in $[\chi_0]$ satisfying the condition

$$(nc\chi' - (n-1)\omega) \wedge \chi'^{n-2} > 0,$$

then there exists a sequence of points and times $(x_i, t_i) \in M \times [0, \infty)$ with $d(x_i, \tilde{S}) \to 0$ and $t_i \to \infty$ such that

$$\left(|\phi| + |\Delta_\omega \phi|\right)(x_i, t_i) \to \infty.$$
$d(x, \bar{S})$ refers to the distance between the point $x$ and the set $\bar{S}$ with respect to the metric $g$.

In two dimensions we will see that the conditions of this theorem are in fact always met. Donaldson [Do1] had noted that on Kähler surfaces, the condition 

$$[nc\chi_0 - \omega] > 0$$

is satisfied for all Kähler classes $[\chi_0]$ and $[\omega]$ if there are no curves of negative self-intersection on $M$. He remarked that if this inequality is violated then one might expect the flow to blow up over some such curves. We confirm this conjecture in the following sense.

**Theorem 1.4** Suppose $n = 2$ and the class $(nc\chi_0 - [\omega])$ is not Kähler. Then there exists a positive integer $m$, irreducible curves of negative self-intersection $E_1, \ldots, E_m$ on $M$ and positive real numbers $a_1, \ldots, a_m$ such that if $D = \sum_{\nu=1}^m a_\nu E_\nu$ and $S = \cup_{\nu=1}^m E_\nu$ then

$$[nc\chi_0 - \omega] - [D] > 0,$$

and we obtain the same estimates (a) and (b) of Theorem 1.3. Moreover, with $\bar{S}$ as in that theorem, there exists a sequence of points and times $(x_i, t_i) \in M \times [0, \infty)$ with $d(x_i, \bar{S}) \to 0$ and $t_i \to \infty$ such that

$$|\phi| + |\nabla \omega \phi|(x_i, t_i) \to \infty.$$

In section 2 we discuss our notation and give some preliminaries about the J-flow, the $I$ and $J$ functionals, and the Mabuchi energy. In section 3 we prove the main estimates of Theorem 1.3. Finally, in section 4 we prove the main results and make a few remarks and conjectures.

**2. Preliminaries**

From now on, we assume that $\omega$ has been scaled so that $c = 1/n$. We will work in local coordinates, and write

$$\omega = \frac{\sqrt{-1}}{2} g_{\bar{z}^i} dz^i \wedge d\bar{z}^\bar{j}, \quad \chi_0 = \frac{\sqrt{-1}}{2} \chi_{0,\bar{j}} dz^i \wedge d\bar{z}^\bar{j},$$

and

$$\chi = \frac{\sqrt{-1}}{2} \chi_{i\bar{j}} dz^i \wedge d\bar{z}^\bar{j} = \frac{\sqrt{-1}}{2} (\chi_{0,\bar{j}} + \partial_i \partial_j \phi) dz^i \wedge d\bar{z}^\bar{j},$$
where \( \chi = \chi_\phi \) (suppressing the \( t \)-subscript.) The operators \( \Lambda_\omega \) and \( \Lambda_\chi \) act on \((1, 1)\) forms \( \alpha = \sqrt{-1} \alpha_{ij} dz^i \wedge dz^j \) by

\[
\Lambda_\omega \alpha = g^{ij} \alpha_{ij}, \quad \text{and} \quad \Lambda_\chi \alpha = \chi^{ij} \alpha_{ij}.
\]

The \( J \)-flow (1.1) can be written

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= \frac{1}{n} (1 - \Lambda_\chi \omega) \quad \text{for} \quad \phi|_{t=0} = \phi_0 \in \mathcal{H}, \\
\end{align*}
\]

(2.1)

Differentiating with respect to \( t \) gives

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t} \right) = \tilde{\Delta} \left( \frac{\partial \phi}{\partial t} \right),
\]

(2.2)

where the operator \( \tilde{\Delta} \) acts on functions \( f \) by

\[
\tilde{\Delta} f = \frac{1}{n} h^{kl} \partial_k \partial_l f,
\]

for

\[
h^{kl} = \chi^{kJ} \chi^{lJ} g_{ij}.
\]

By the maximum principle for parabolic equations, (2.2) implies that

\[
\inf_M (\Lambda_\chi \omega_0) \leq \Lambda_\chi \omega \leq \sup_M (\Lambda_\chi \omega_0),
\]

which gives a lower bound for \( \chi \),

\[
\chi \geq \frac{1}{\sup_M (\Lambda_\chi \omega_0)} \omega.
\]

(2.3)

We will now define some important functionals on the space \( \mathcal{H} \). The \( J \)-functional [Ch2] is defined by

\[
J(\phi) = J_{\omega, \chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \omega \wedge \chi_{\phi_t}^{n-1} \frac{1}{(n-1)!} dt,
\]

where \( \{\phi_t\} \) is a path in \( \mathcal{H} \) between \( 0 \) and \( \phi \). The functional is independent of the choice of path. The \( I \)-functional is a well-known functional (see [Ma1]) given by

\[
I(\phi) = I_{\chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \chi_{\phi_t}^n \frac{1}{n!} dt.
\]
It will also be convenient to define a normalized $J$-functional which we will denote by $\hat{J} = \hat{J}_{\omega, \chi_0}$, given by

$$\hat{J}(\phi) = J(\phi) - ncI(\phi) = \int_0^1 \int_M \frac{\partial \phi_i}{\partial t} (\omega \wedge \chi_0^{n-1} - c\chi_0^n) \frac{dt}{(n-1)!}.$$

Note that $\hat{J}$ has the property $\hat{J}(\phi + C) = \hat{J}(\phi)$ for constants $C$.

The $J$-flow is the gradient flow of the functional $\hat{J}$ on $\mathcal{H}$. Alternatively, as in [Ch2], one could normalize using the $I$ functional, and consider the space $\mathcal{H}_0 = \{ \phi \in \mathcal{H} \mid I(\phi) = 0 \}$. In that case, the $J$-flow is the gradient flow of $J$.

We will now describe the relationship between $\hat{J}$ and the Mabuchi energy. Under the assumption that $\omega = -\text{Ric}(\chi_0) > 0$, Chen’s formula [Ch2] for the Mabuchi energy can be written

$$M_{\chi_0}(\phi) = \int_M \log \left( \frac{\chi_0^n \phi}{\chi_0^n} \right) \frac{\chi_0^n}{n!} + \hat{J}_{\omega, \chi_0}(\phi).$$

(2.4)

It is easy to see that the first term is bounded below. In fact, we will see in section 4 that it is proper.

Let us now define what is meant by the ‘properness’ of a functional on $\mathcal{H}$. First, recall the definitions of the Aubin-Yau energy functionals [Au2] which are also called $I$ and $J$ in the literature. To avoid confusion we will denote them by $I^E$ and $J^E$. They are given by

$$I^E_{\chi_0}(\phi) = \frac{\sqrt{-1}}{2n!V} \sum_{i=0}^{n-1} \int_M \partial \phi \wedge \overline{\partial} \phi \wedge \chi_0^i \wedge \chi_0^{n-1-i}$$

$$J^E_{\chi_0}(\phi) = \frac{\sqrt{-1}}{2n!V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial \phi \wedge \overline{\partial} \phi \wedge \chi_0^i \wedge \chi_0^{n-1-i},$$

for $V = \int_M \chi_0^n/n!$, and they satisfy

$$\frac{1}{n+1}I^E_{\chi_0} \leq J^E_{\chi_0} \leq \frac{n}{n+1}I^E_{\chi_0}.$$

Following Tian [Ti3], we say that a functional $T$ on $\mathcal{H}$ is proper if there exists an increasing function $f : [0, \infty) \to \mathbb{R}$, satisfying $f(x) \to \infty$ as $x \to \infty$, such that for any $\phi \in \mathcal{H}$,

$$T(\phi) \geq f \left( J^E_{\chi_0}(\phi) \right).$$
In the course of the proofs, \( C_1, C_2, \ldots \) will denote uniform constants. Curvature expressions such as \( R_{klij} \) refer to the metric \( g_{ij} \).

3. Estimates on \( \phi \) and \( \Lambda_\omega \chi \)

In this section, we prove the main estimates (a) and (b) of Theorem 1.3. We assume that \( c = 1/n \). As in the statement of the theorem, let \( s_\nu \), for \( \nu = 1, \ldots, m \), be holomorphic sections of the line bundles associated to the divisors \( S_\nu \) which vanish on \( S_\nu \) and let \( h_\nu \) be hermitian metrics on these line bundles. Then since \( \chi' \in [\chi_0] - [D] \), there exists a smooth function \( \theta \) such that

\[
\chi' = \chi_0 + \sum_{\nu=1}^m \frac{\sqrt{-1}}{2\pi} a_\nu \partial \bar{\partial} \log h_\nu + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \theta.
\]

Replacing \( h_1 \) with \( h_1 e^{-\theta\pi/a_1} \), we may assume that \( \theta = 0 \). The change in \( h_1 \) will only modify the constants \( C \) and \( C' \) in the final estimates.

The condition for \( \chi' \) gives us, for \( \epsilon > 0 \) sufficiently small,

\[
(\chi' - (n-1)\omega) \wedge \chi'^{m-2} > 2\epsilon \chi'^{m-1}. \tag{3.1}
\]

For \( \phi \) in \( \mathcal{H} \), set

\[
\hat{\phi} = \phi - \sum_{\nu=1}^m \frac{a_\nu}{\pi} \log |s_\nu|^2_{h_\nu}.
\]

Then, for any smooth \( \phi \), the corresponding \( \hat{\phi} \) is a smooth function on \( M - S \), and

\[
\chi' = \chi_\phi - \frac{\sqrt{-1}}{2} \partial \bar{\partial} \hat{\phi}.
\]

We use the maximum principle to prove the following estimate on the second derivatives of \( \phi \).

**Lemma 3.1** Let \( \phi = \phi_t \) be a solution of the J-flow (1.1) on \( [0, \infty) \), with the assumptions of Theorem 1.3. Then there exist positive constants \( A \) and \( C_1 \) depending only on \( \omega, \chi_0, \chi' \) and \( \phi_0 \) such that for any time \( t \geq 0 \), \( \chi = \chi_{\phi_t} \) satisfies

\[
\Lambda_{\omega} \chi \leq C_1 e^{A(\hat{\phi} - \inf_{(M-S) \times [0, t]} \hat{\phi})}, \tag{3.2}
\]

on \( M - S \).
Proof First note that the infimum of $\hat{\phi}$ on $M - S$ exists because $\hat{\phi}(x)$ tends to infinity as $x$ approaches $S$. Choose the constant $A$ to be large enough so that

$$- \frac{1}{A(\Lambda_\omega \chi)}(h^{kl} R_{kl} \tilde{\chi}_{ij} - \chi^{kl} R_{kl}) \leq \epsilon,$$

(3.3)

for $\epsilon > 0$ satisfying (3.1). Note that we can find such a constant by (2.3).

We will calculate the evolution of

$$\log(\Lambda_\omega \chi) - A \hat{\phi},$$
on $M - S$. From [We1], p. 954, we have

$$(\tilde{\Delta} - \frac{\partial}{\partial t}) \log(\Lambda_\omega \chi) \geq \frac{1}{n(\Lambda_\omega \chi)}(h^{kl} R_{kl} \tilde{\chi}_{ij} - \chi^{kl} R_{kl}).$$

Calculate on $M - S$,

$$\begin{align*}
(\tilde{\Delta} - \frac{\partial}{\partial t}) \hat{\phi} &= \frac{1}{n}(h^{kl} \partial_k \hat{\phi} + \chi^{ij} g_{ij} - 1) \\
&= \frac{1}{n}(\chi^{ij} \hat{\phi} + \chi^{ij} g_{ij} - \chi^{kl} R_{kl}) \\
&= \frac{1}{n}(2 \chi^{ij} g_{ij} - h^{kl} \chi^{ij} - 1).
\end{align*}$$

Now consider the point $(x_0, t_0) \in (M - S) \times [0, t]$ at which the quantity $(\log(\Lambda_\omega \chi) - A \hat{\phi})$ achieves its maximum. Note that since $\hat{\phi}(x)$ tends to infinity as $x$ approaches $S$, the maximum is achieved on this set. We may assume that $t_0 > 0$, for if $t_0 = 0$, the estimate follows trivially. At $(x_0, t_0)$, we have

$$0 \geq (\tilde{\Delta} - \frac{\partial}{\partial t})(\log(\Lambda_\omega \chi) - A \hat{\phi})$$

$$\geq \frac{1}{n} \left( \frac{1}{(\Lambda_\omega \chi)}(h^{kl} R_{kl} \tilde{\chi}_{ij} - \chi^{kl} R_{kl}) + A h^{kl} \tilde{\chi}_{ij} - 2A \chi^{ij} g_{ij} + A \right).$$

Hence at $(x_0, t_0)$,

$$1 + h^{kl} \tilde{\chi}_{kl} - 2 \chi^{ij} g_{ij} \leq - \frac{1}{A(\Lambda_\omega \chi)}(h^{kl} R_{kl} \tilde{\chi}_{ij} - \chi^{kl} R_{kl})$$

$$\leq \epsilon,$$

using (3.3). We now compute in normal coordinates for the metric $\chi_{ij}'$ so that the metric $\chi_{ij}'$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. The metric $g_{ij}$ may not be diagonal in this basis, but we will denote its (positive) diagonal entries $g_{ii}$ by $\mu_i$. 

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The above inequality becomes
\[ 1 + \sum_{i=1}^{n} \frac{\mu_i}{\lambda_i} - 2 \sum_{i=1}^{n} \frac{\mu_i}{\lambda_i} \leq \epsilon. \]

Fix \( k \) between 1 and \( n \). Completing the square, we obtain
\[ 1 + \sum_{i=1 \atop i \neq k}^{n} \mu_i \left( \frac{1}{\lambda_i} - 1 \right)^2 - \sum_{i=1 \atop i \neq k}^{n} \mu_i + \frac{\mu_k}{\lambda_k^2} - 2 \frac{\mu_k}{\lambda_k} \leq \epsilon. \]

Now we will make use of the condition (3.1). Writing \( \beta_k \) for the (1,1) form \( \sqrt{-1} dz^k \wedge dz^{\bar{k}} \), we have
\[(\chi' - (n-1)\omega) \wedge \chi^{n-2} \wedge \beta_k > 2\epsilon \chi^{n-1} \wedge \beta_k.\]

Writing \( \chi' \) and \( \omega \) in our coordinates, this inequality becomes
\[(n-1)! \beta_1 \wedge \cdots \wedge \beta_n - (n-1)! \sum_{i=1 \atop i \neq k}^{n} \mu_i \beta_1 \wedge \cdots \wedge \beta_n > 2\epsilon (n-1)! \beta_1 \wedge \cdots \beta_n,\]
and hence
\[1 - \sum_{i=1 \atop i \neq k}^{n} \mu_i > 2\epsilon.\]

Returning to (3.4) this gives us
\[2\epsilon - 2 \frac{\mu_k}{\lambda_k} < 1 + \sum_{i=1 \atop i \neq k}^{n} \mu_i \left( \frac{1}{\lambda_i} - 1 \right)^2 - \sum_{i=1 \atop i \neq k}^{n} \mu_i + \frac{\mu_k}{\lambda_k^2} - 2 \frac{\mu_k}{\lambda_k} \leq \epsilon,\]
from which we obtain
\[\frac{\lambda_k}{\mu_k} < \frac{2}{\epsilon},\]
for \( k = 1, \ldots, n \). Summing in \( k \) we obtain the estimate
\[\Lambda_{\omega \chi} \leq C_1 = \frac{2n}{\epsilon}\]
at the point \((x_0, t_0)\). Then on on \((M - S) \times [0, t]\), we have
\[\log(\Lambda_{\omega \chi}) - \hat{A} \hat{\phi} \leq \log C_1 - A \inf_{(M - S) \times [0, t]} \hat{\phi}.\]

Exponentiating gives
\[\Lambda_{\omega \chi} \leq C_1 e^{A(\hat{\phi} - \inf_{(M - S) \times [0, t]} \hat{\phi})},\]
completing the proof of Lemma 3.1.
We will now prove the zero order estimate for $\phi$.

**Lemma 3.2** There exists a constant $C$ such that the solution $\phi = \phi_t$ of the $J$-flow (1.1) satisfies

$$\hat{\phi}_t(x) = \phi_t(x) - \sum_{\nu=1}^{m} \frac{a_{\nu}}{\pi} \log |s_{\nu}|^2_{\text{Res}}(x) \geq -C, \text{ for all } x \in M - S.$$ 

**Proof** Fix a time $t$ and choose $t_0$ in the interval $[0, t]$ so that

$$\inf_{M - S} \hat{\phi}_{t_0} = \inf_{(M - S) \times [0, t]} \hat{\phi} = \inf_{(M - S) \times [0, t_0]} \hat{\phi}.$$ 

Define a function $\psi$ on $M - S$ by

$$\psi = \hat{\phi}_{t_0} - \sup_{M} \phi_{t_0}.$$ 

We will prove that $\psi$ is uniformly bounded from below. We will make use of Lemma 3.1. Set

$$u = e^{-B\psi}, \quad \text{for } B = \frac{A}{1 - \delta},$$

where $A$ is the constant from Lemma 3.1 and $\delta > 0$ is a small positive constant to be determined. Then $u$ is a smooth non-negative function, which we will show is uniformly bounded from above. We have the following lemma.

**Lemma 3.3** For any $p \geq 1$,

$$\int_{M} |\nabla u^{p/2}|^2 \frac{\omega^n}{n!} \leq C_2 p \| u \|_{C^0}^{1 - \delta} \int_{M} u^{p-(1-\delta)} \frac{\omega^n}{n!}, \quad (3.5)$$

**Proof** The proof is a modification of the argument given in [We2]. First note that although $|\nabla u^{p/2}|^2$ may not be smooth, it is integrable. For any small, positive $\eta$, let $T_\eta$ be a tubular neighborhood of $S$ of radius $\eta$ (with respect to the metric $g$.) Then calculate

$$\int_{M - T_\eta} |\nabla u^{p/2}|^2 \frac{\omega^n}{n!} = \sqrt{-1} \int_{M - T_\eta} \partial e^{-B\psi} \wedge \overline{\partial} e^{-B\psi} \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{B^2 p^2}{4} \sqrt{-1} \int_{M - T_\eta} e^{-B\psi} \partial \psi \wedge \overline{\partial} \psi \wedge \frac{\omega^{n-1}}{(n-1)!} = -

\frac{Bp}{4} \sqrt{-1} \int_{M - T_\eta} \partial (e^{-B\psi}) \wedge \overline{\partial} \psi \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{Bp}{2} \int_{M - T_\eta} e^{-B\psi} \frac{\sqrt{-1}}{2} \partial \overline{\partial} \psi \wedge \frac{\omega^{n-1}}{(n-1)!} + E_\eta.$$
where $E_\eta$ is the boundary term,

$$E_\eta = -\frac{Bp}{4} \sqrt{-1} \int_{\partial(M-T_\eta)} e^{-Bp\psi} \bar{\partial}\psi \wedge \frac{\omega^{n-1}}{(n-1)!},$$

obtained by integrating by parts. Note that $E_\eta \to 0$ as $\eta \to 0$. Now from the definition of $\psi$,

$$\int_{M-T_\eta} |\nabla u_p|_{2}^2 \omega_n = \frac{Bp}{2} \int_{M-T_\eta} e^{-Bp\psi} (\chi\phi_0 - \chi') \wedge \frac{\omega^{n-1}}{(n-1)!} + E_\eta$$

$$\leq \frac{Bp}{2} \int_{M-T_\eta} e^{-Bp\psi} (\Lambda\omega \chi \phi_0) \frac{\omega^n}{n!} + E_\eta$$

$$\leq \frac{C_1 Bp}{2} \int_{M-T_\eta} e^{-Bp\psi} e^{A(\psi - \inf_{M-S} \psi)} \frac{\omega^n}{n!} + E_\eta$$

$$= \frac{C_1 Bp}{2} e^{-A \inf_{M-S} \psi} \int_{M-T_\eta} e^{-B(\phi_0 - \sup T_0)} \chi_0 \frac{\omega^n}{n!} + E_\eta$$

where, in the third line, we have used the estimate

$$\Lambda \omega \chi \leq C_1 e^{A(\phi_0 - \inf_{M-S} \phi)} = C_1 e^{A(\psi - \inf_{M-S} \psi)},$$

of Theorem 3.1. Letting $\eta \to 0$ completes the proof.

We now use the following lemma from [We2], which we quote without proof.

**Lemma 3.4** If $u \geq 0$ satisfies the estimate (3.5) for all $p \geq 1$, then for some constant $C_3$ independent of $u$,

$$\|u\|_{C^0} \leq C_3 \left( \int_M u^d \omega^n \right)^{1/d}. $$

To obtain the upper bound for $u$, observe that

$$\int_M u^d \omega^n = \int_M |s_1|^{2B\delta a_1 / \pi} |s_m|^{2B\delta a_m / \pi} e^{-B\delta(\phi_0 - \sup T_0)} \chi_0 \frac{\omega^n}{n!}$$

$$\leq C_4 \int_M e^{-B\delta(\phi_0 - \sup T_0) \chi_0} \frac{\omega^n}{n!}.$$

Choosing $\delta$ small enough we can bound the right hand side. This is due to the following proposition of Tian [Ti1], based on a result of Hörmander [Hö] (it was used by Tian to define the $\alpha$-invariant - see [TiYa, So] for more on this important invariant.)

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Proposition 3.5 There exists $\alpha > 0$ and $C_5$ depending only on $(M, \chi_0)$ such that
\[ \int_M e^{-\alpha \phi} \frac{\chi_0^n}{n!} \leq C_5, \] for all $\phi \in C^2(M)$ satisfying
\[ \chi_0 + \frac{\sqrt{-1}}{2} \partial \overline{\partial} \phi > 0, \quad \sup_M \phi = 0. \]

Hence we obtain an upper bound for $u$, and so a lower bound for $\psi$. To get the lower bound for $\hat{\phi}$, we require the following lemma from [We2].

Lemma 3.6 Let $\phi_t$ be a solution of the $J$-flow. There exist positive constants $C_6$ and $C_7$ depending only on the initial data such that
\[ -C_6 \leq \sup_M \phi_t \leq C_6 - C_7 \inf_M \phi_t. \]

Using the first inequality we obtain at time $t$
\[ \inf_{M-S} \hat{\phi}_t \geq \inf_{M-S} \hat{\phi}_{t_0} = \inf_{M-S} \psi + \sup_M \phi_{t_0} \geq -C_8, \]
completing the proof of the lower bound of $\hat{\phi}$ and giving (a). Estimate (b) follows immediately from (a) and Lemma 3.1.

4. Proofs of the main results

In section 3 we proved the estimates (a) and (b) of Theorem 1.3. We will use these to complete the proofs of the remaining theorems.

Proof of Theorem 1.1. (ii) implies (iii) is trivial. To see that (iii) implies (i), suppose that $\chi$ is a solution of the critical equation and choose a normal coordinate system for $g_{ij}$ in which $\chi_{ij}$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. Assuming that $c = 1/n$, we have, as explained in section 1, the inequality
\[ \sum_{i=1}^n \frac{1}{\lambda_i} < 1 \quad \text{for } k = 1, \ldots, n. \]
We will now calculate the form \((\chi - (n-1)\omega) \wedge \chi^{n-2}\). As in section 3, write \(\beta_k\) for the \((1,1)\)-form \(\sqrt{-1}dz^k \wedge d\bar{z}^k\). Then
\[
\chi^{n-1} = (\lambda_1 \beta_1 + \cdots + \lambda_n \beta_n)^{n-1}
\]
\[
= (n-1)! \sum_{k=1}^{n} \lambda_1 \cdots \hat{\lambda}_k \cdots \lambda_n \beta_1 \wedge \cdots \wedge \hat{\beta}_k \wedge \cdots \wedge \beta_n,
\]
where ‘\(^\wedge\)’ indicates that the symbol should be omitted. Similarly,
\[
\omega \wedge \chi^{n-2} = (\beta_1 + \cdots + \beta_n) \wedge (\lambda_1 \beta_1 + \cdots + \lambda_n \beta_n)^{n-2}
\]
\[
= (n-2)! \sum_{k=1}^{n} \sum_{i=1}^{n} \lambda_1 \cdots \hat{\lambda}_i \cdots \hat{\lambda}_k \cdots \lambda_n \beta_1 \wedge \cdots \wedge \hat{\beta}_i \wedge \cdots \wedge \beta_n.
\]
Then the condition
\[
(\chi - (n-1)\omega) \wedge \chi^{n-2} > 0,
\]
is equivalent to
\[
\lambda_1 \cdots \hat{\lambda}_k \cdots \lambda_n - \sum_{i=1}^{n} \lambda_1 \cdots \hat{\lambda}_i \cdots \hat{\lambda}_k \cdots \lambda_n > 0, \quad \text{for } k = 1, \ldots, n,
\]
which, since the \(\lambda_i\) are positive, is precisely our inequality
\[
\sum_{i=1}^{n} \frac{1}{\lambda_i} < 1 \quad \text{for } k = 1, \ldots, n.
\]

To show that (i) implies (ii) we will use the estimates (a) and (b) of Theorem 1.3. Let \(D\) be the zero divisor and let \(s\) be equal to the constant section 1. Then estimate (a) gives us a uniform lower bound for \(\phi\) along the flow. We can apply Lemma 3.6 to obtain a uniform upper bound for \(\phi\). Estimate (b) gives us a uniform bound on \(\Lambda \omega \chi \phi\) and hence on the second derivatives \(\partial_i \partial_j \phi\). Bounds on all the derivatives of \(\phi\) and the convergence to a critical metric then follow by the same arguments as in [Ch3, We1, We2]. This completes the proof.

**Proof of Theorem 1.2.** Arguing as in [Ch2], we can apply Yau’s Theorem [Ya1] and assume that \(-\omega = \text{Ric}(\chi_0)\). Since the conditions of Theorem 1.1 are satisfied, the \(J\)-flow converges to a smooth critical metric. Since the
$J$-flow is the gradient flow for $\tilde{J}_{\omega,\chi}$ and the critical metrics are unique it follows that $\tilde{J}_{\omega,\chi}$ is bounded below (alternatively, apply Proposition 3 of [Ch2].) Then from (2.4), we have

$$M_{\chi_0}(\phi) \geq \int_M \log \left( \frac{\chi_n^\phi}{\chi_0^n} \right) \frac{\chi_n^\phi}{n!} - C_1.$$  

The theorem then follows immediately from a lemma due to Tian [Ti4] and the properties of $I_{\chi_0}^E$ and $J_{\chi_0}^E$.

**Lemma 4.1** There exist positive constants $\alpha$ and $C_3$ such that

$$\frac{1}{V} \int_M \log \left( \frac{\chi_n^\phi}{\chi_0^n} \right) \frac{\chi_n^\phi}{n!} \geq \alpha I_{\chi_0}^E(\phi) - C_3.$$  

**Proof** The result can be found in [Ti4], p.95 (it is assumed there that $c_1(M)$ is a multiple of $[\chi_0]$, but this plays no part in the proof.) For the reader’s convenience, we will give the proof here. From Proposition 3.5 there exist positive constants $C_2$ and $\alpha$ such that

$$\frac{1}{V} \int_M e^{-\log \frac{\chi_n^\phi}{\chi_0^n} - \alpha (\phi - \sup_M \phi) \chi_0^n} \frac{\chi_n^\phi}{n!} = \frac{1}{V} \int_M e^{-\alpha (\phi - \sup_M \phi) \chi_0^n} \frac{\chi_n^\phi}{n!} \leq C_2.$$  

By the convexity of the exponential function, we have

$$\frac{1}{V} \int_M \log \left( \frac{\chi_n^\phi}{\chi_0^n} \right) \frac{\chi_n^\phi}{n!} \geq -\frac{\alpha}{V} \int_M (\phi - \sup \phi) \frac{\chi_n^\phi}{n!} - \log C_2.$$  

Writing $I_{\chi_0}^E$ in the form

$$I_{\chi_0}^E(\phi) = \frac{1}{n!V} \int_M \phi (\chi_n^\phi - \chi_0^n),$$  

completes the proof of the lemma and hence the theorem.

**Remark 4.2** From Tian’s conjecture [Ti4] on the equivalence of the existence of a cscK metric and the properness of the Mabuchi energy, we would expect that there exists a cscK metric in each class $[\chi_0]$ in the set $V$ described in Theorem 1.2 (c.f. [We3], Conjecture 5.2.1.) Moreover, on the boundary of $V$, where $\chi'$ and $\omega$ satisfy

$$\left( -n \frac{\pi c_1(M) \cdot [\chi_0]^n}{[\chi_0]^n} \chi' - (n-1)\omega \right) \wedge \chi'^{n-2} \geq 0,$$  

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we would expect the Mabuchi energy to be bounded below. Also, one would expect the classes in $V$ to be $K$-stable and those on the boundary to be $K$-semistable [Ti3].

**Proof of Theorem 1.3.** It is only left to prove the last statement of this theorem. Suppose for a contradiction that there exists some $\eta > 0$ such that

$$(|\phi| + |\nabla_\omega \phi|)(x) \leq C_4, \quad \text{for } x \in T_\eta,$$

where $T_\eta$ is a tubular neighborhood of $\tilde{S}$ of radius $\eta$. We will show this implies that the J-flow converges to a critical metric, contradicting our assumption that there does not exist $\chi' \in [\chi_0]$ satisfying (1.6).

First, it is not difficult to see that there exist a finite number of divisors $D^{(1)}, \ldots, D^{(k)}$ in the linear system $|D|$ such that any $x$ in $M - T_\eta$ is at least a distance $\eta/2$ from one of the $D^{(i)}$.

We claim that there exists $C_5$ such that for $i = 1, \ldots, k$,

$$\inf_{M - S^{(i)}} \hat{\phi}(i) \leq C_5,$$

where we are using the obvious notation. To see this, pick for each $i$ a point $x^{(i)}$ in $T_\eta$ which does not lie on $S^{(i)}$. Then

$$\inf_{M - S^{(i)}} \hat{\phi}(i) \leq \phi(x^{(i)}) - \sum_{\nu=1}^{m^{(i)}} \frac{a^{(i)}_{\nu}}{\pi} \log \left| s^{(i)}_{\nu}(x^{(i)}) \right|^2 h^{(i)}_{\nu}(x^{(i)}),$$

which is uniformly bounded. Then from estimate (b) of Theorem 1.3, we have for $i = 1, \ldots, k$,

$$\Lambda_\omega \chi \leq C_6 e^{A(\hat{\phi}(i) - \inf_{M - S^{(i)}} \hat{\phi}(i))}.$$

Note that this is a stronger estimate than (3.2), and enables us to do the following. Define $\psi^{(i)} = \hat{\phi}(i) - \sup_M \phi$ as before. Then we have at any time $t$,

$$\Lambda_\omega \chi \leq C_6 e^{A(\psi^{(i)} - \inf_{M - S^{(i)}} \psi^{(i)})}.$$

By the same argument as in section 3, we obtain a uniform constant $C_7$ such that

$$\psi^{(i)} \geq -C_7.$$
Hence on $M - S(i)$,

\[
\Lambda\omega\chi \leq C_6 e^{A(\psi(i) - \inf_{M - S(i)} \psi(i))} \\
\leq C_8 e^{A\psi(i)} \\
= C_8 e^{A(-\sum_{\nu=1}^m a_{\nu}(i) \log |s_{\nu}(i)|^2_{h_{\nu}(i)} + \phi - \sup_M \phi)} \\
\leq \frac{C_8}{|s_1(i)|^{2Aa_1(i)/\pi} \cdots |s_m(i)|^{2Aa_m(i)/\pi} h_{m(i)}^{\nu} h_{m(i)}^{\nu}}.
\]

Then we see that $\Lambda\omega\chi$ is uniformly bounded on $M - \overline{T}_\eta$ and therefore on $M$. Arguing now as in the smooth case, we obtain a uniform bound on $\phi$ and so the J-flow converges to a critical metric, giving us the contradiction.

**Remark 4.3** It is not difficult to see that the set $\tilde{S}$ of Theorem 1.3 is non-empty unless (1.6) is satisfied for some metric $\chi'$ in $[\chi_0]$. Otherwise, by arguments similar to those above it can be shown that $\phi$ must tend to infinity on the whole of $M$, contradicting Lemma 3.6.

**Remark 4.4** It would be interesting to know whether one can improve on these estimates (for example, by showing that $\sup \phi$ is bounded, at least away from the singular set.) We conjecture that in the boundary case, when there exists a metric $\chi'$ in $[\chi_0]$ satisfying

\[
(nc\chi' - (n - 1)\omega) \wedge \chi'^{n-2} \geq 0,
\]

the J-flow converges to a critical metric on compact subsets outside the singular set $S$.

**Proof of Theorem 1.4.** This theorem follows almost immediately from the following proposition.

**Proposition 4.5** Let $M$ be a Kähler surface with a Kähler class $\beta$ in $H^{1,1}(M, \mathbb{R})$. If $\alpha$ in $H^{1,1}(M, \mathbb{R})$ satisfies

\[
\alpha^2 > 0 \quad \text{and} \quad \alpha \cdot \beta > 0
\]

then either $\alpha$ is a Kähler class or there exists a positive integer $m$, curves of negative self intersection $E_1, \ldots, E_m$ and positive real numbers $a_1, \ldots, a_m$ such that

\[
\alpha - \sum_{\nu=1}^m a_{\nu}[E_{\nu}]
\]

is a Kähler class.
Proof This result is essentially contained in [La] (see also [Bu]) and so we will just give an outline of the proof here. By Lemma 5.2 and Theorem 5.1 of [La], the conditions $\alpha^2 > 0$ and $\alpha \cdot \beta > 0$ imply the existence of a Kähler current $\tau$ such that $\alpha = [\tau]$. That is, $\tau$ is a closed (1,1) current satisfying $\tau \geq \psi$ for some strictly positive smooth (1,1) form $\psi$. By Siu’s decomposition [Si] and a result of Demailly [De], there exist constants $c_\nu \geq 0$ and irreducible curves $D_\nu$ such that

$$\tau = \tau + \sum_{\nu=1}^{\infty} c_\nu D_\nu,$$

where $\tau$ is a Kähler current which is smooth away from a finite number of points. By a smoothing argument, the class $[\tau]$ is Kähler. Write for each $\nu$,

$$c_\nu D_\nu = a_\nu E_\nu + b_\nu C_\nu,$$

where the $E_\nu$ and $C_\nu$ are irreducible curves satisfying $E_\nu^2 \leq 0$ and $C_\nu^2 \geq 0$ and $a_\nu$ and $b_\nu$ are nonnegative constants. We have

$$[\tau - \sum_{\nu=1}^{m} a_\nu E_\nu] = [\tau + \sum_{\nu=1}^{\infty} b_\nu C_\nu] + [\epsilon_m],$$

where $\epsilon_m$ is the current

$$\epsilon_m = \sum_{\nu=m+1}^{\infty} a_\nu E_\nu,$$

which tends to zero (in the weak topology of currents) as $m$ tends to infinity. Now the $C_\nu$ are nef and the Kähler cone is stable under the addition of nef classes [La, Bu] so the class $[\tau + \sum_{\nu=1}^{\infty} b_\nu C_\nu]$ is Kähler. Since the Kähler cone is open, there exists $m$ large enough such that

$$[\tau - \sum_{\nu=1}^{m} a_\nu E_\nu] = \alpha - \sum_{\nu=1}^{m} a_\nu [E_\nu]$$

is Kähler.

We can apply this proposition in our case to $\alpha = [nc\chi_0 - \omega]$, since

$$[nc\chi_0 - \omega]^2 = [\omega]^2 > 0,$$

and

$$[nc\chi_0 - \omega] \cdot [\chi_0] = [\omega] \cdot [\chi_0] > 0.$$
Remark 4.6 Can Theorem 1.4 be generalized to higher dimensions?

Remark 4.7 If a surface $M$ with $c_1(M) < 0$ has no curves of negative self-intersection then from Theorem 1.2 we see that the Mabuchi energy is proper for any class. It would be interesting to see examples of such manifolds where cscK metrics can be constructed in those classes away from the canonical class (c.f. [Fi].) If there do exist curves of negative self intersection, then one might guess that they form obstructions to the stability of $(M, [\chi_0])$ in some sense (this could be related to the slope stability of [Ro], [RoTh].)

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