1 Introduction

The global holomorphic invariant $\alpha_G(X)$ introduced by Tian [14], Tian and Yau [13] is closely related to the existence of Kähler-Einstein metrics. In his solution to the Calabi conjecture, Yau [20] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with nonpositive first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there exist known obstructions such as the Futaki invariant. For a compact Kähler manifold $X$ with positive first Chern class, Tian [14] proved that $X$ admits a Kähler-Einstein metric if $\alpha_G(X) > \frac{n}{n+1}$, where $n = \dim X$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $\mathbb{CP}^2 \# 1 \mathbb{CP}^2$ and $\mathbb{CP}^2 \# 2 \mathbb{CP}^2$ [16].

There have been many nice results on the classification of toric Fano manifolds. Mabuchi discovered that if a toric Fano manifold is Kähler-Einstein then the barycenter of the polyhedron defined by its anticanonical divisor is at the origin. V. Batyrev and E. Selivanova [2] gave a lower bound of $\alpha$-invariant of symmetric toric Fano manifolds which is sufficient to show the existence of a Kähler-Einstein metric.

In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman kernel on polarized Kähler metrics to approximate almost plurisubharmonic functions and obtain a formula to calculate the $\alpha_G$-invariants of all toric Fano manifolds precisely. This formula improves the result of V. Batyrev and E. Selivanova [2] and generalize the earlier results [12] on the estimates of $\alpha$ invariants on $\mathbb{CP}^2 \# 1 \mathbb{CP}^2$ and $\mathbb{CP}^2 \# 2 \mathbb{CP}^2$.

First let us recall the definition of $\alpha_G(X)$. Let $X$ be an $n$-dimensional compact complex manifold with positive first Chern class $c_1(X)$ and $G$ a compact subgroup of $Aut(X)$. Choose a $G$-invariant Kähler metric $g = g_{ij}$ on $X$ such that $\omega = \frac{\sqrt{-1}}{2\pi} \sum g_{ij} dz_i \wedge d\bar{z}_j \in c_1(X)$. 

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**Definition** Let $P_G(X, \omega)$ be the set of all $C^2$-smooth $G$-invariant real-valued functions $\varphi$ such that $\sup_X \varphi = 0$ and $\omega + \frac{i}{2\pi} \partial \overline{\partial} \varphi > 0$. The $\alpha_G(X)$ invariant is defined as supemum of all $\alpha > 0$ such that there exists $C(\alpha)$ with

$$\int_X e^{-\alpha \varphi} \omega^n \leq C(\alpha)$$

for all $\varphi \in P_G(X, \omega)$.

If $X$ is a toric Fano $n$-fold, the anticanonical imbedding induces a natural moment map whose image is a convex polyhedron $\Sigma$. The maximal torus $T \subset Aut(X)$ acting on $X$ has an open dense orbit $U \subset X$. We denote by $N(T) \subset Aut(X)$ the normalizer of $T$ in $Aut(X)$, then $W(X) = N(T)/T$ acts on $\Sigma$ as a finite linear group.

**Definition** Let $S = \{ v \in \Sigma \mid gv = v \text{ for all } g \in W(X) \}$ be the set of all fixed points of $\Sigma$ by $W(X)$. We call $X$ symmetric if $S = \{0\}$. If $S \neq \{0\}$ then for any $0 \neq v \in S$, we define $w_v$ related with $v$ by $w_v = \partial \Sigma \cap \{-tv \mid t \geq 0\}$. We also choose $G \subset Aut(X)$ to be the group generated by $(S^1)^n$ and $W(X)$.

Our main theorem is

**Theorem 1.1** Let $X$ be a toric Fano manifold of complex dimension $n$. Then

(a) $\alpha_G(X) = 1$ if $X$ is symmetric, otherwise

(b) $\alpha_G(X) = \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}} \leq \frac{1}{2}$.

**Corollary 1.1** Let $X$ be a toric Fano manifold. Then $X$ is symmetric if and only if $\alpha_G(X) = 1$.

Tian also introduced the invariants $\alpha_{m,G}$ analogous to $\alpha_G$. Define $P_{m,G}(X, \omega) = \{ \varphi \in C^\infty(X, \mathbb{R}) \mid \sup_X \varphi = 0, \varphi \text{ is } G\text{-invariant and there exists a basis } \{ S^m_i \}_{0 \leq i \leq d_m - 1} \text{ of } H^0(X, K_X^{-m}) \text{ such that } \omega + \frac{i}{2\pi} \partial \overline{\partial} \varphi = \frac{1}{m} \partial \overline{\partial} \log(\sum_{i=0}^{d_m-1} |S^m_i|^2) \}$, where $d_m = \dim H^0(X, K_X^{-m})$ and $m$ is large. Here the almost plurisubharmonic functions $\varphi$ are restricted to the almost plurisubharmonic functions generated by $H^0(X, \mathcal{O}(K_X^{-m}))$. We then define for $m$ large, $\alpha_{m,G}(X) = \sup \{ \alpha \mid \text{there exists } C(\alpha) > 0 \text{ such that for all } \varphi \in P_{m,G}(X), \int_X e^{-\alpha \varphi} dV \leq C(\alpha) \}$. These invariants reflect the singularities of the divisors cut out by the global sections of $K_X^{-m}$. Tian [15] also raised the question whether $\alpha_{m,G} = \alpha_G$ for $m$ large enough. This question is related to the ascending chain condition, which is itself related to the finiteness of certain flip-flop procedures in the minimal model program in algebraic geometry. We can confirm the question in the case of toric Fano manifolds.
Theorem 1.2 If $X$ is a toric Fano manifold, then $\{\alpha_{m,G}(X)\}_{m \geq 1}$ is stationary. More precisely, $\alpha_{m,G}(X) = \alpha_G(X)$ if $m \geq m_0$, where $m_0$ is the least positive integer such that $m_0v$ is an integral point and $v$ is a rational point and a minimizer of $\min_{0 \neq v \in S} \frac{|w_v|}{|v|}$.

Acknowledgements. The author deeply thanks his advisor, Professor D.H. Phong for his constant encouragement and help. He also thanks Professor Jacob Sturk and Professor Zhiqin Lu for their suggestion on this work. This paper is part of the author’s future Ph.D. thesis in Math Department of Columbia University.

2 Toric Fano manifolds

Let $N$ be a lattice of rank $n$, $M = Hom(N, \mathbb{Z})$ the dual lattice. $M_R = M \otimes \mathbb{Z} R$, $N_R = N \otimes \mathbb{Z} R$. Let $X = X_\Sigma$ be a smooth projective toric $n$-fold defined by a complete fan $\Delta$ of regular cones $\Delta \subset M_R$ and denote $\Delta(i)$ the $i$-dimensional cone of $\Delta$. We put $T = \mathbb{C}^* = \{(t_1, t_2, ..., t_n) | t_i \in \mathbb{C}^*\}$. For $a \in M$ and $b \in N$, we define $< a, b > \in \mathbb{Z}$, $\chi^a \in Hom_{algp}(T, \mathbb{C}^*)$ by

$$< a, b > = \sum_{i=1}^{n} a_i b_i,$$
$$\chi^a((t_1, ..., t_n)) = t_1^{a_1} t_2^{a_2} ... t_n^{a_n}.$$

For each $\rho \in \Delta(1)$, let $b_\rho$ denote the unique fundamental generator of $\rho$. We now consider the divisor $K = -\sum_{\rho \in \Delta(1)} D(\rho)$ on $X = X_\Delta$. The following theorem is due to Demazure[4].

Theorem 2.1 $K$ is a canonical divisor of $X_\Delta$, and the following are equivalent:

(a) $X_\Delta$ is a toric fano manifold.
(b) $K_X^{-1}$ is ample.
(c) $K_X^{-1}$ is very ample.
(d) $\Sigma_{K_X^{-1}} = \{ a \in M_R | < a, b_\rho > \leq 1 \text{ for all } \rho \in \Delta(1) \}$ is an $n$-dimensional compact convex polyhedron whose vertices are exactly $\{ a_\tau | \tau \in \Delta(n) \}$, where each $a_\tau$ denotes the unique element of $M$ such that $< a_\tau, b > = 1$ for all fundamental generators $b$ of $\tau$.

The maximal torus $T \subset Aut(X)$ acting on $X$ has an open dense orbit $U \subset X$, so the normalizer $N(T) \subset Aut(X)$ of $T$ has a natural action on $U$. Let $W(X) = N(T)/T$ and
we identify the maximal torus $T \subset \text{Aut}(X)$ with an open dense orbit $U$ in $X$ by choosing an arbitrary point $x_0 \in U$, then we have the following splitting short exact sequence

$$1 \to T \to N(T) \to W(X) \to 1,$$

which gives an embedding $W(X) \hookrightarrow N(T)$. Denote by $K(T) = (S^1)^n$ the maximal compact subgroup in $T$. We choose $G$ to be the maximal compact subgroup in $N(T)$ generated by $W(X)$ and $K(T)$, so that we have the short exact sequence

$$1 \to K(T) \to G \to W(X) \to 1.$$

**Proposition 2.1** Let $X=X_\Delta$ be a smooth projective toric n-fold defined by a complete regular polyhedral fan $\Delta$. Then the group $W(X)$ is isomorphic to the finite group of all symmetries of $\Delta$, i.e., $W(X)$ is isomorphic to a subgroup of $\text{GL}(N)(\simeq \text{GL}(n,\mathbb{Z}))$ consisting of all elements $\gamma \in \text{GL}(N)$ such that $\gamma(\Delta) = \Delta$.

**Remark** $W(X)$ is as well isomorphic to a subgroup of $\text{GL}(M)(\simeq \text{GL}(n,\mathbb{Z}))$ consisting of all elements $\gamma \in \text{GL}(M)$ such that $\gamma(\Sigma) = \Sigma$.

**Definition** A toric n-fold $X$ is symmetric if the trivial character is the only $W(X)$-invariant algebraic character of $T$, i.e. $N^{W(X)} = \{\chi \in N | g\chi = \chi \text{ for all } g \in W(X)\} = \{0\}$.

### 3 Holomorphic approximation of almost plurisubharmonic functions

In this section, we will employ the technique in [15, 21] to obtain the approximation of almost plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [21].

**Theorem 3.1** (Zelditch) Let $M$ be a compact complex manifold of dimension $n$ and let $(L, h) \to M$ be a positive hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega = \text{Ric}(h)$. For each $m \in N$, $h$ induces a hermitian
metric $h_m$ on $L^m$. Let $\{S^n_0, S^n_1, ..., S^n_{d_m-1}\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!}\omega^n$ is the volume form of $g$. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} ||S^m_i(x)||_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + ...$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any $k$:

$$\left|\sum_{i=0}^{d_m-1} ||S^m_i(x)||_{h_m}^2 - \sum_{j<k} a_j(x)m^{n-j}\right|_{C^k} \leq C_{R,k} m^{n-R} \quad (3.1)$$

where $C_{R,k}$ depends on $R, k$ and the manifold $M$.

For any $\varphi \in P(M, \omega)$ let

$$\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi > 0$$

$$\tilde{h} = \h e^{-m\varphi}.$$

Let $\tilde{h}_m$ be the induced hermitian metric of $\tilde{h}$ on $L^m$, $\{\tilde{S}^m_0, \tilde{S}^m_1, ..., \tilde{S}^m_{d_m-1}\}$ be any orthonormal basis of $H^0(M, L^m)$ with respect to the inner product

$$(S_1, S_2)_{\tilde{h}_m} = \int_M \tilde{h}_m(S_1(x), S_2(x))dV_{\tilde{\omega}}.$$

By the definition of $\tilde{h}_m$, we have

$$\sum_{i=0}^{d_m-1} ||\tilde{S}^m_i(x)||_{\tilde{h}_m}^2 = \left(\sum_{i=0}^{d_m-1} ||S^m_i(x)||_{h_m}^2 \right) e^{-m\varphi}.$$

Thus

$$\varphi - \frac{1}{m} \log \left(\sum_{i=0}^{d_m-1} ||\tilde{S}^m_i(x)||_{\tilde{h}_m}^2 \right) = -\frac{1}{m} \log \left(\sum_{i=0}^{d_m-1} ||S^m_i(x)||_{h_m}^2 \right).$$

For any positive integer $R$, by Theorem 3.1 we have

$$\frac{1}{m} \log \left(\sum_{j<R} \tilde{a}_j(x)m^{n-j} \right) = \frac{1}{m} \log m^n \left(\sum_{j<R} \tilde{a}_j(x)m^{n-j} \right)$$

$$= \frac{n}{m} \log m + \frac{1}{m} \log(1 + O\left(\frac{1}{m}\right)) \to 0.$$
if we let \( m \to +\infty \).

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

**Corollary 3.1**

\[
\| \varphi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} \| \tilde{S}_i^m(x) \|_{b_m}^2 \right) \|_{C^k} \to 0, \text{ as } m \to +\infty.
\]

In other words, any almost plurisubharmonic function can be approximated by the logarithms of holomorphic sections of \( L^m \).

### 4 Proof of The Main Theorems

Suppose \( X_\Delta \) is Fano, then one obtains a convex \( W(X) \)-invariant polyhedron \( \Sigma \) in \( M_\mathbb{R} \) defined by \( \Sigma = \{ a \in M_\mathbb{R} \mid < a, b_\rho > \leq 1, \text{ for all } \rho \in \Delta(1) \} \) where \( b_\rho \) is the fundamental generator of \( \rho \). Let \( L(\Sigma) = \{ v_0, v_1, ..., v_k \} = M \cap \Sigma \) be the set of all lattice points in \( \Sigma \). Then \( v_0, v_1, ..., v_k \) determine algebraic characters \( \chi_i : T \to \mathbb{C}^* \) of \( T(\rho=0, 1, ..., k) \).

Moreover, we have
\[
|\chi_i(x)|^2 = e^{<v_i, y>}, \quad i = 0, ..., k,
\]
(4.2)

where \( y \) is the image of \( x \) under the canonical projection \( \pi : T \to M_\mathbb{R} \). Let us define \( u : U \to \mathbb{R} \) as follows:
\[
u = \log(\sum_{i=0}^{k} |\chi_i(x)|^2), \quad x \subset U \simeq T.
\]

Since \( u \) is \( K(T) \)-invariant, \( u \) descends to a function \( \tilde{u} : M_\mathbb{R} \to \mathbb{R} \) defined by
\[
\tilde{u} = \log(\sum_{i=0}^{k} e^{<v_i, y>}), \quad y \subset M_\mathbb{R}.
\]

Consider the \( G \)-invariant hermitian metric \( g = g_{ij} \) on \( X \) such that the restriction of the corresponding to \( g \) differential 2-form on \( U \) is defined by
\[
\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u.
\]

The metric \( g \) is exactly the pull-back of Fubini-Study metric from \( P^m \) with respect to the anticanonical embedding \( X \hookrightarrow P^m \) defined by the algebraic characters \( \chi_0, \chi_1, ..., \chi_k \).
Let \( \Sigma^{(m)} = \{ a \in M \mathbb{R} \mid < a, b_p > \leq m \) and \( L(\Sigma^{(m)}) = \{ v_0, ..., v_{k_m} - 1 \} = M \cap \Sigma^{(m)} \), where \( k_m = \dim H^0(X, O(K_X^{-m})) \) and \( \chi^\mu: T \to \mathbb{C}^* \) defined by \( |\chi^\mu(x)|^2 = e^{<\mu, y>} \). We have the following lemma (see [7] p66).

**Lemma 4.1** \( H^0(X, O(K_X^{-m})) = \oplus_{\mu \in L(\Sigma^{(m)})} C \cdot \chi^\mu \).

**Proposition 4.1** \( \{ \chi^\mu \}_{\mu \in L(\Sigma^{(m)})} \) is an orthogonal basis of \( H^0(X, O(K_X^{-m})) \) with respect to the inner product \( <, >_{h_m} \), where the hermitian metric of \( K_X^{-m} \) is defined by \( h^m = \frac{1}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} \).

**Proof** The inner product of \( \chi^\mu \) and \( \chi^{\nu} \) is

\[
\int_X \langle \chi^\mu, \chi^{\nu} \rangle_{h_m} = \int_T \frac{1}{(\sum_{i=0}^k |\chi_i|^m)^m} \omega^n = \int_T \frac{|z_1|^{\mu_1 + v_i} ... |z_n|^{\mu_n + v_i} e^{(\mu_1 - v_i)\theta_1} ... e^{(\mu_n - v_i)\theta_n}}{(\sum_{i=0}^k |\chi_i|^m)^m} \omega^n,
\]

which is 0 if \( \mu \neq \nu \).

For any \( \varphi \in P_G(X, \omega) \), it induces a hermitian metric on \( K^{-1} \) by \( \tilde{h} = he^{-\varphi} \). Denote by \( \Sigma^{(m)} = \{ mv \mid v \in \Sigma \} \) the dilation of \( \Sigma \) by \( m \) times. Then by Corollary 3.1

\[
\varphi(x) = \lim_{m \to \infty} \frac{1}{m} \log \frac{\sum_{\mu \in L(\Sigma^{(m)})} a^{(m)}_{\mu} |\chi^\mu(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m}, \tag{4.3}
\]

where \( a^{(m)}_{\mu} = \frac{1}{|\chi^\mu(x)|^2_{h,m}} \).

**Lemma 4.2** There exists \( \epsilon > 0 \) such that for any \( \varphi \in P_G(X, \omega) \) and \( \tilde{m} > 0 \) there exist \( m > \tilde{m} \) and \( \mu \in L(\Sigma^{(m)}) \) with \( (a^{(m)}_{\mu})_{h,m} > \epsilon \).

**Proof** Otherwise, for any \( \epsilon > 0 \) there exists \( \varphi_\epsilon \) and \( \tilde{m} \) such that for any \( m > \tilde{m} \) and \( \mu \in L(\Sigma^{(m)}) \) we have \( (a^{(m)}_{\mu})_{h,m} < \epsilon \). By choosing \( m \) large enough we have

\[
\varphi_\epsilon(x) \leq \frac{1}{m} \log \left( \sum_{\mu \in L(\Sigma^{(m)})} \frac{|\chi^\mu(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} + \log \epsilon \right) = \frac{1}{m} \log \left( \sum_{\mu \in L(\Sigma^{(m)})} \frac{|\chi^\mu(x)|^2}{(\sum_{i=0}^k |\chi_i(x)|^2)^m} \right) + \log \epsilon \leq \frac{1}{m} \log \left( \sum_{\mu \in L(\Sigma^{(m)})} 1 \right) + \log \epsilon \leq \text{Const} + \log \epsilon.
\]
Since \( \epsilon \) can be chosen arbitrarily small, the above inequality implies that \( \varphi_\epsilon \to -\infty \) uniformly as \( \epsilon \) goes to 0, which contradicts the fact that \( \sup_X \varphi = 0 \).

Now we can prove a lower bound for any \( \varphi \in P_G(X, \omega) \). By Lemma 4.2 we have

\[
\varphi(x) = \lim_{m \to \infty} \frac{1}{m} \log \frac{\sum_{\mu \in L(S^{(m)})} a_{\mu} |\chi(x)|^2}{(\sum_{k=0}^{m} |\chi(x)|^2)^m}
\]

(4.4)

\[
\geq \frac{1}{m} \log \frac{\sum_{\mu \in W(X)} |\chi^{g\mu}(x)|^2}{(\sum_{k=0}^{m} |\chi(x)|^2)^m} - C_1
\]

(4.5)

\[
\geq \log \frac{|\chi(x)|}{(\sum_{k=0}^{m} |\chi(x)|^2)} - C_1
\]

(4.6)

\[
\geq \log \frac{|\chi(x)|}{(\sum_{k=0}^{m} |\chi(x)|^2)} - C_1
\]

(4.7)

for some \( \mu \in L(S^{(m)}) \).

Put \( y_i = \log |t_i|^2 \) \( t_i = e^{\frac{y_i^2}{2} + \sqrt{-1} \theta_i} \), then

\[
\frac{dt_i}{t_i} = \frac{1}{2} dy_i + \sqrt{-1} d\theta_i
\]

\[
\frac{dt_i}{|t_i|^2} = \frac{1}{2} dy_i + \sqrt{-1} d\theta_i
\]

\[
\frac{dt_i \wedge dt_j}{|t_i|^2} = -\sqrt{-1} dt_i \wedge d\theta_i
\]

\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u = \sum_{i,j} \frac{\partial^2 \bar{u}}{\partial y_i \partial y_j} \frac{dt_i \wedge dt_j}{t_i t_j}
\]

\[
(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u)^n = \det(\frac{\partial^2 \bar{u}}{\partial y_i \partial y_j})dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n.
\]

Any \( \varphi \in P_G(X, \omega) \) descends to a function \( \tilde{\varphi} : M_R \to \mathbb{R} \). Put \( v = \sum_{\mu \in W(X)} g\mu \), then we have

\[
\tilde{\varphi}(y) \geq \log \frac{e^{<v, y>}}{\sum_{i=0}^{m} e^{<v, y>}} - C_1.
\]

(4.8)

**Lemma 4.3** Let \( \tilde{F} = e^{\tilde{u}} \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} \), then \( 0 < c \leq \tilde{F} \leq C \).

**Proof** \( e^{-u \frac{dt_1 \wedge dt_1 \wedge ... \wedge dt_n \wedge d\theta}{|t_1|^2 ... |t_n|^2}} = e^{-\tilde{u}} dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n \) can be extended to a non-vanishing volume form on \( X \). Also

\[
(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u)^n = \det \frac{\partial^2 u}{\partial t_i \partial t_j} dt_1 \wedge ... \wedge dt_n \wedge d\theta_n
\]

\[
= \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n
\]

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is a non-vanishing volume form, so the quotient of these two volume form must be positive and bounded. This proves the lemma.

Now we are ready to prove Theorem 1.1. For any \(0 < \alpha < 1\),

\[
\int_X e^{-\alpha \varphi} \omega^n = \int_X e^{-\alpha \varphi} \left( \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} u \right)^n \tag{4.9}
\]

\[
= \int_{\mathbb{R}^n} e^{-\alpha \varphi} \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} dy_1...dy_n \tag{4.10}
\]

\[
\leq C_2 \int_{\mathbb{R}^n} e^{-\alpha \varphi - \tilde{u}} dy_1...dy_n \tag{4.11}
\]

\[
\leq C_3 \int_{\mathbb{R}^n} e^{-\alpha \log \sum e^{v_i,y} - \log (\sum e^{v_i,y})} dy_1...dy_n \tag{4.12}
\]

\[
= C_3 \int_{\mathbb{R}^n} e^{-\alpha <v,y>} (\sum e^{v_i,y}) \frac{1}{1-\alpha} dy_1...dy_n, \tag{4.13}
\]

here \(v\) is some fixed point of \(\Sigma\) by \(W(X)\).

If \(S = \{0\}\), i.e. \(X\) is symmetric, then \(v = 0\) and therefore we have

\[
\int_X e^{-\alpha \varphi} \omega^n \leq C_3 \int_{\mathbb{R}^n} \frac{1}{(\sum_{i=0}^k e^{v_i,y})^{1-\alpha}} dy_1...dy_n \leq C(\alpha)
\]

for all \(\varphi\) because every \(n\)-dimensional cone \(\sigma_j \in \Delta(j = 1, ..., l)\) is generated by a basis of the lattice \(N\) and \(N_R = \sigma_1 \cup ... \cup \sigma_l\). This implies \(\alpha_G(X) \geq 1\).

**Remark** By Tian’s theorem [14] \(X\) admits Kähler-Einstein metric. This is also obtained by V. Batyrev and E.N. Selivanova [2] using different techniques.

If \(S \neq \{0\}\) then for any \(0 \neq v \in S\), we have \(w_v \in \partial \Sigma\) related with \(v\) by

\[
<v_v, v> = -|w_v||v|.
\]

For fixed \(\alpha < 1\) the integral

\[
\int_{\mathbb{R}^n} \frac{e^{-\alpha <v,y>}}{(\sum e^{v_i,y})^{1-\alpha}} dy_1...dy_n \tag{4.14}
\]

\[
= \int_{\mathbb{R}^n} \frac{e^{-\frac{\alpha}{1-\alpha} <v,y>}}{(\sum e^{v_i,y})^{1-\alpha}} dy_1...dy_n \tag{4.15}
\]

\[
= \int_{\mathbb{R}^n} \frac{1}{(\sum e^{v_i,y})^{\frac{\alpha}{1-\alpha}}} dy_1...dy_n \tag{4.16}
\]

is finite if

\[-\frac{\alpha}{1-\alpha} v \in \text{int}(\Sigma),\]
\[ < - \frac{\alpha}{1 - \alpha} v, w_v > = \frac{\alpha}{1 - \alpha} |v| |w_v| \leq |w_v|^2. \]

Then for all \( \alpha < \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \) the integral \( \int_X e^{-\alpha \varphi} \omega^n \) is uniformly bounded from above for all \( \varphi \). Therefore

\[ \alpha_G(X) \geq \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}}. \quad (4.17) \]

In order to estimate the upper bound of \( \alpha_G(X) \), we will construct a sequence of almost plurisubharmonic functions. Suppose \( S \neq \{0\} \), then for all \( \alpha \) with \( 1 > \alpha > \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \) we choose \( \tilde{\varphi} = \log(\sum_{\epsilon \in \Gamma} e^\epsilon \epsilon^{\varphi}) \) which is increasing and uniformly bounded from above, where \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \) is achieved at \( \tilde{v} \in S \). Then by Fatou’s lemma we have

\[ \lim_{\epsilon \to 0} \int_X e^{-\alpha \tilde{\varphi} \epsilon} \omega^n = \int_X e^{-\alpha \tilde{\varphi} \epsilon} \omega^n = \infty. \]

This implies \( \alpha_G(X) \leq \frac{\min_{0 \neq v \in S} \frac{|w_v|}{|v|}}{1 + \min_{0 \neq v \in S} \frac{|w_v|}{|v|}}. \)

Combining the above estimates together, we have proved the Theorem 1.1.

Also it’s obvious to see that \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \leq 1 \) for non-symmetric toric Fano manifold \( X \) thus \( \alpha_G(X) \leq \frac{1}{2} \). This shows that there does not exist any non-symmetric toric Fano manifold such that its \( \alpha_G \)-invariant is greater than \( \frac{n}{n+1} \), which is a sufficient condition for the existence of a Kähler-Einstein metric.

Now we prove Theorem 1.2 by making use of the proof of Theorem 1.1.

It is easy to see \( \alpha_{m,G}(X) \) is decreasing as \( m \) goes to the infinity. By the argument of the upper bound for the \( \alpha_G \)-invariant, we can directly have the following corollary which answers the question proposed by Tian[15] in the special case of toric Fano manifolds.

**Corollary 4.1** If \( X \) is a toric Fano manifold, then \( \{\alpha_{m,G}(X)\}_m \) is decreasing and stationary. More precisely, \( \alpha_{m,G}(X) = \alpha_G(X) \) if \( m \geq m_0 \), where \( m_0 \) is the least positive integer such that \( m_0 v \in M \) and \( v \) is the minimizer of \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \).

**Proof** It suffices to show that there exists rational points \( v \) and \( w_v \) which minimize \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \). Let \( L \) be a simplex of the least dimension in \( \partial \Sigma \), which contains either \( v \) or \( w_v \) which is a minimizer of \( \min_{0 \neq v \in S} \frac{|w_v|}{|v|} \). We can assume \( v \in L \). If \( \dim L = 0 \), then we are done since \( L \) is a vertex of \( \Sigma \) and therefore is a rational point. If \( \dim L > 0 \), \( v \) is an interior
point of $L$ and let $L'$ be the simplex of $\partial \Sigma$ which contains $w_v$ corresponding to $v$ and has the least dimension. If $\overrightarrow{vw_v}$ is the only line joining $L, L'$ and 0, then both $v$ and $w_v$ are rational points since they are determined by linear equations with rational coefficients. If there exists another line joining $L, L'$ and 0, then it is easy to show that there exists a proper simplex of $L$ with its dimension less than $L$ and it contains a minimizer of $\min_{0 \neq v \in S} \frac{|uv_v|}{|v|}$, which contradicts the assumption of the least dimension of $L$. Therefore we prove the corollary.

**Remark** Wang and Zhu [19] recently proved that the vanishing of the Futaki invariant implies the existence of Kähler-Einstein metrics on toric Fano manifolds. It would be interesting to show that the vanishing of the Futaki invariant implies K-stability for toric Fano manifolds. Also it would be very interesting to know if there exists any nonsymmetric toric Fano manifold which is Kähler-Einstein.

**Remark** The $\alpha$-invariant is closely related to the method of multiplier ideal sheaf developed by Nadel[9] and Siu[11]. Both methods give sufficient conditions ensuring the existence of a Kähler-Einstein metric with the presence of a sufficiently large group of automorphisms. The strategy employed by Nadel to construct Kähler-Einstein metrics is to rule out the existence of any $G$-invariant subscheme with certain properties. In the case of non-symmetric toric Fano n-folds, we can always find a $G$-invariant $\varphi \in L^1_{\text{loc}}(X)$ such that $\omega + \frac{1}{2\pi} \partial \overline{\partial} \varphi$ is a positive current and for some $\frac{n}{n+1} < \alpha < 1$

\[
\int_X e^{-\alpha \varphi} \omega^n = \infty
\]

due to the fact that $\alpha_G(X) \leq \frac{1}{2}$. Therefore we can always construct those $G$-invariant subschemes in Nadel’s condition, which is associated by the multiplier ideal sheaf $\mathcal{J}(\varphi)$.

## 5 Examples

In this sections we will calculate the $\alpha$ invariants for 2-dimensional toric Fano manifolds. Here (1) (2) (3) (4) correspond to $\mathbb{CP}^2$ and $\mathbb{CP}^2$ blow-up at 1, 2 and 3 points and (5) (6) (7) (8) are the corresponding polyhedrons $\Sigma$.
\( \mathbb{CP}^2 \) and \( \mathbb{CP}^2 \) blow-up at 3 points are symmetric thus their \( \alpha_G \)-invariants are both equal to 1.

For \( \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \), its fixed points by \( G \) on the boundary of the polyhedron in (6) are \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( \left( -\frac{1}{2}, -\frac{1}{2} \right) \). Since \( \frac{|(1/2,1/2)|}{|-1/2,-1/2|} = 1 \), it is easy to see \( \alpha_G(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) = \frac{1}{2} \).

For \( \mathbb{CP}^2 \# 2 \overline{\mathbb{CP}^2} \) its fixed points by \( G \) on the boundary of the polyhedron in (7) are \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( (-1, -1) \). Since \( \frac{|(1/2,1/2)|}{|(-1,-1)|} = \frac{1}{2} \), it is easy to see \( \alpha_G(\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}^2}) = \frac{1}{3} \).

The above calculation confirms our earlier results in [12].
References


