PSEUDO-REPRESENTATIONS OF WEIGHT ONE

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Abstract. We prove that the determinant (generalized pseudocharacter) associated to the Hecke algebra of Katz modular forms of weight one and level prime to p is unramified at p.

Contents

1. Introduction

Let p be prime, and let $N \geq 5$ be an integer prime to p. Let $X_1(N)$ be the modular curve considered as a smooth proper curve over $\mathrm{Spec}(\mathbf{Z}_p)$, and let ω be the pushforward of the relative dualizing sheaf from the universal elliptic curve $\mathcal{E}/X_1(N)$. For general m, one knows that the map:

$$H^0(X_1(N),\omega) \rightarrow H^0(X_1(N),\omega/p^m\omega)$$

need not be surjective. In particular, if $\mathbf{T}_1$ denotes the $\mathbf{Z}_p$-subalgebra of

$$\mathrm{End}(\varinjlim H^0(X_1(N),\omega/p^m\omega)) = \mathrm{End}(H^0(X_1(N),\omega\otimes\mathbf{Q}_p/\mathbf{Z}_p)) \simeq \mathrm{End}(H^1(X_1(N),\omega(-\infty)),$$

generated by the Hecke operators T_n and $\langle n\rangle$ for $(n,N)=1$, then $\mathbf{T}_1$ may be bigger than the classical Hecke algebra acting on $H^0(X_1(N),\omega)$. Our main theorem is as follows:

Theorem 1.1. *Let $\mathbf{T}_1$ be the $\mathbf{Z}_p$-subalgebra of $\mathrm{End}_{\mathbf{Z}_p}(H^0(X_1(N),\omega\otimes\mathbf{Q}_p/\mathbf{Z}_p))$ generated by the Hecke operators T_n and $\langle n\rangle$ for all n prime to N. There is a determinant:*

$$D_1 : \mathbf{T}_1[G_{\mathbf{Q}}] \rightarrow \mathbf{T}_1,$$

of degree 2, which is unramified[1] *at all l prime to N, including $l=p$, such that the characteristic polynomial of any Frobenius element at l equals:*

$$P_{D_1,\mathrm{Frob}_l}(X) = 1 - T_l X + \langle l\rangle X^2.$$

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[1] If $G_{\mathbf{Q},S}$ denotes the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside the set S of places consisting of ∞ and the primes dividing N, then the condition that D_1 is unramified outside N is equivalent to saying that D_1 factors through $\mathbf{T}_1[G_{\mathbf{Q},S}]$.

The Hecke alegebra $\mathbf{T}_1$ is a semi-local ring. If $\mathfrak{m} \subset \mathbf{T}_1$ is a maximal ideal, then the base change of D_1 to $\mathbf{T}_1/\mathfrak{m}$ arises from a semi-simple Galois representation $\overline{\rho}$ valued in the algebraic closure of $\mathbf{T}_1/\mathfrak{m}$ [Che14, Theorem 2.12]. If this representation is irreducible, then, by a theorem of Carayol [Car94], the base change of D_1 to the localization $\mathbf{T}_\mathfrak{m} := (\mathbf{T}_1)_\mathfrak{m}$ arises from a representation $\rho : G_\mathbf{Q} \to \mathrm{GL}_2(\mathbf{T}_\mathfrak{m})$. Our theorem shows this representation is unramified at p. For $p > 2$, this is a consequence of Theorem 3.11 of [CG]. Hence the main interest of our result is to residually reducible representations. However, the result is new even for absolutely irreducible representations when $p = 2$ (although there are significant partial results by Wiese [Wie14]). Although the proof of Theorem 1.1 is similar to that of Theorem 3.11 of [CG], it is more direct, and does not rely on any explicit analysis of the ordinary deformation rings of Snowden [Sno]. Hence this paper can also be seen as providing a simplification of the proof of Theorem 3.11 of *ibid*.

Part of the content of this note is that we will give the general definition of what it means for a determinant to be *ordinary*. If $V/\overline{\mathbf{Q}}_p$ is an n dimensional representation of $G_\mathbf{Q}$ and $\mathbf{w} := (w_1, \ldots w_n)$ is a non-decreasing sequence of integers, we say that V is ordinary of weight $\mathbf{w}$, if there exists a complete $G_{\mathbf{Q}_p}$-stable flag on V such that the action of the inertia group at p on the i-th component of the associated graded representation is through the w_i-th power of the cyclotomic character. In Definition 2.2.1, we say what it means for a determinant to be ordinary of weight $\mathbf{w}$. By Theorem 2.2.3, ordinary, $\overline{\mathbf{Q}}_p$-valued determinants are in bijection with finite-dimensional, semi-simple p-adic $G_\mathbf{Q}$-representations which are ordinary in the classical sense.

Carl Wang Erickson [WE, §7.3] has given a definition of ordinariness for 2-dimensional pseudo-representations (determinants) under an assumption of locally p-distinguishedness (see also [WWE15, Definition 3.4.1] and [CV03, §3]). We expect — but have not checked — that our definition agrees with his in this setting. However, for our application, it is important that we be able to work in non-p-distinguished situations; the definition we give accommodates for this. It would also be interesting to check, in the context where the associated residual determinant is associated to a globally irreducible representation $\overline{\rho}$, whether our definition of ordinary coincides with the purely local condition given by David Geraghty [Ger].

The construction of the determinant $D_1 : \mathbf{T}_1[G_\mathbf{Q}] \to \mathbf{T}_1$ without any condition at p is a standard application of p-adic interpolation. There is a determinant valued in a weight one p-adic Hecke algebra which is constructed by interpolating the determinants attached to modular Galois representations in cohomological weight. The determinant D_1 is obtained from this determinant via specialization.

The main content of Theorem 1.1 is that D_1 is unramified at p. In section 2.3, we will establish a general criterion to check if an *ordinary determinant* of degree n and weight $\mathbf{0}$ is unramified at p. If V is a p-distinguished representation and $\phi \in G_\mathbf{Q}$ is a Frobenius element at p, then any ordinary flag on V gives rise to a complete ordering of the roots of the characteristic polynomial of ϕ. The representation V is unramified if and only if every possible ordering can be obtained from some ordinary flag. Our criteria for unramifiedness (Theorem 2.3.1) is an appropriate generalization of this equivalence beyond the p-distinguished case. The proof that D_1 is unramified will proceed by first showing D_1 is *ordinary*, and then showing that D_1 satisfies this criterion.

While, in this note, we will be interested in *ordinary* determinants, our constructions equally apply to any determinant which is *upper triangular* and has some fixed *graded inertial* *representation.* In the final section, we discuss the behavior of D_1 for the primes that exactly divide N. At semi-stable level, we show the full weight-one Hecke algebra is the quotient of (and conjecturally isomorphic to) a certain pseudo-deformation ring with extra structure.

2. Determinants

Let G be a group and A be a ring. If M is a free, finite rank A-module equipped with a linear G action, then one may consider the family of characteristic polynomials associated to the elements of G. This family is a strong invariant of a representation. For example, if A is an an algebraically closed field, the family of characteristic polynomials determines the representation uniquely up to semi-simplification.

In this section, we recall the notion of a *determinant* as given by Gaëtan Chenevier [Che14, pg. 3]. The polynomials that occur in a family of characteristic polynomials are highly interdependent. Informally, one may regard a *determinant* as a family of polynomials, which satisfy many of those relations that occur in a family of characteristic polynomials. Rather than attempt to describe the relations in this family directly, the insight behind the definition of a *determinant* is that one can enlarge such a family to include certain elements that generalize the characteristic polynomial's construction, and then use these cousins to the characteristic polynomials to succinctly express relations in the original family.

The characteristic polynomial of an element $g \in G$ is by definition the determinant of the endomorphism $X - g$ acting on $M \otimes_A A[X]$. To enlarge the family of characteristic polynomials, Chenevier records, for every A-algebra B and every element $x \in B[G]$, the determinant of x acting on $M \otimes_A B$. This enlarged family can be organized as a series of set theoretic maps $det : B[G] \to B$, one for each A-algebra B, which satisfy the following compatibilities:

(1) the maps det are natural in B,
(2) $\det(1) = 1$ and the element $\det(xy) = \det(x)\det(y)$ for all $x, y \in B[G]$,
(3) and $\det(bx) = b^n \det(x)$, where $b \in B$ and n is equal to the rank of M.

A *determinant* is simply a family of maps which are compatible in these three ways.

Definition 2.0.1. *Let A be a ring*[2]*, G be a topological group, and n be a positive integer. A degree n determinant is a continuous A-valued polynomial law*[3] *$D : A[G] \to A$, which is multiplicative and homogeneous of degree d. If B is an A-algebra and $m \in B[G]$, we call $P_{D,m}(X) := D(1 - mX) \in A[X]$ the characteristic polynomial of m.*

Enlarging a family of characteristic polynomials attached to a representation to a *determinant* is a superficial procedure. There is a formula of Shimshon Amitsur, which expresses the classical determinant of the sum of two elements in a matrix algebra (over any ring) in terms of the coefficients in the characteristic polynomials of various products [Ami80, Theorems A and B, pg. 179,182]. Given a family of characteristic polynomials $P_{D,g}$, with $g \in G$, one

[2]All rings considered in this note will carry a Hausdorff topfology, and, with the exception of group rings, will be commutative. Our terminology will suppress these topological and algebraic considerations. We use the terms *module* and *algebra* to denote a Hausdorff topological module and a commutative, Hausdorff topological algebra, respectively.

[3]An A-valued polynomial law between two A-modules M and N is by definition a natural transformation $N \otimes_A B \to M \otimes_A B$ on the category of commutative A-algebras B. A polynomial law is called multiplicative if $D(1) = 1$ and $D(xy) = D(x)D(y)$ for all $x, y \in A[G] \otimes B$, and is called homogeneous of degree d, if $D(xb) = b^d D(x)$ for all $x \in A[G] \otimes B$ and $b \in B$. A polynomial law is called continuous if its characteristic polynomial map on G given by $g \mapsto P_{D,g}$ is continuous.

may use Amistur's formulae to recover the full determinant. The only function of enlarging a family of characteristic polynomials to a determinant is to express certain relations in the original family simply and elegantly.

A general determinant has a similar anatomy. First, given any A-algebra B, one may show for each $m \in B[G]$,

$$P_{D,m}(X) = 1 + \sum_{i=1}^{d} (-1)^i c_i(m) X^i,$$

where $c_i(m) \in B$ and d is the degree of D. Each coefficient map $m \mapsto c_i(m)$ defines an A-valued homogeneous polynomial law of degree i. The coefficient c_i satisfies Amistur's formula (see formula 2.1), and using this formula one may show that a determinant is uniquely determined by its set of characteristic polynomials $P_{D,g}$, with $g \in G$.

Given a determinant $D : A[G] \to A$, one may specialize D to the category of algebras over any fixed A-algebra B. The result is a determinant $D_B : B[G] \to B$. If F is an algebraically closed field, then [Che14, Theorem 2.12] states that all F-valued determinants arise from a unique semi-simple representation of G. The determinant D, therefore, provides a means to associate to any geometric point of $\mathrm{Spec}(A)$ a semi-simple representation of G; to the point $\pi : A \to F$, one associates the representation with determinant $D_\pi := D_F$. The determinant D interpolates between these specializations. In light of this, one may regard $\mathrm{Spec}(A)$ as coarsely parameterizing a family of semi-simple representations of G – the parametrization and specific representations being determined by D.

If one thinks of $\mathrm{Spec}(A)$ in this way, it is natural to try to define sub-loci, which coarsely parameterize subfamilies of representations with certain properties. If P is a property of n-dimensional representations of G, then we say P can be *interpolated*, if for any determinant $D : A[G] \to A$ there exists a closed subscheme $\mathrm{Spec}(A^P) \subset \mathrm{Spec}(A)$, which is natural in A and whose geometric points consist exactly of the geometric points of $\mathrm{Spec}(A)$ with property P. The endofunctor $(A, D) \mapsto (A^P, D_{A^P})$ is called an interpolation of P. In general, a property may admit many interpolations. In what follows, we will choose specific interpolations for the properties we study. We will say D has property P, if $A^P = A$.

Let $G_{\mathbf{Q}}$ be the absolute Galois group of $\mathbf{Q}$. In this note, we are interested in the local properties at p of the determinants on $G_{\mathbf{Q}}$ valued in the p-adic completions of weight one Hecke algebras. The p-adic Galois representation attached to any classical weight one Eigenform of level prime to p is unramified at p. In the next section, we describe an interpolation of "unramifiedness at p" due to Chenevier. The goal of this note will be to show that the degree 2 determinants attached to weight one Hecke algebras are unramified at p.

2.1. **Unramified Determinants.** Let G be any (topological) group and V/F be a finite dimensional representation of G. Let $h \in G$. What relations occur in the family of characteristic polynomials attached to G by its action on V, if the element h acts by the identity on V?

Above, we saw that relations in a family of characteristic polynomials can be more easily expressed by extending that family to a determinant. The same holds here. If B be an F-algebra and h lies in the kernel of a representation V, then $h - 1$ acts by 0 on $V \otimes B$. So for any $b \in B$ and $y \in B[G]$, the endomorphism $b(h-1) + y$ acts by y on $V \otimes_F B$. It follows $\det(b(h-1) + y) = \det(y)$. The converse is true if V is an irreducible representation over an algebraically closed field. In this case, if $x \in F[G]$ and $\det(bx + y) = \det(y)$ for all F-algebras B and elements $b \in B$ and $y \in B[G]$, then x acts by 0.

For a general determinant, Chenevier defines the *kernel* as:

Definition 2.1.1. [Che14, Section 1.17] *Let G be a topological group, A be a ring, and $D : A[G] \to A$ be a determinant. The kernel of D is the subset $ker(P) \subset A[G]$, consisting of those elements x such that for any A-algebra B and elements $b \in B$ and $y \in B[G]$,*

$$D(bx + y) = D(y).$$

If D arises from a semi-simple representation $\rho : G \to GL_n(F)$ valued in an algebraically closed field, then $h \in G$ lies in $\ker(\rho)$, if and only if $1 - h \in \ker(D)$. For a determinant with coefficients in a general commutative ring, lemma 1.9 of [Che14] states that the kernel of a determinant is a two sided ideal of $A[G]$, and consists of those elements $m \in A[G]$ such that for any A-algebra B, and element $x \in B[G]$, the characteristic polynomial $P_{D,mx}(X) = 1$. In fact, as the next lemma shows, to verify if $m \in \ker(D)$, it is enough to check $P_{D,mx}(X) = 1$ only for those elements $x \in A[G]$.

Lemma 2.1.2. *Let $D : A[G] \to A$ be a determinant. An element $m \in A[G]$ lies in $\ker(D)$ if and only if $P_{D,mx}(X) = 1$ for all $x \in A[G]$.*

To prove this lemma, we will use Amistur's formulae.

Digression: Amistur's Fomulae. Amistur's formulae express the value of the characteristic polynomial coefficients of a sum of elements in terms of the characteristic polynomial coefficients of certian products of those elements. The specific products are Lyndon words on the elements. We begin by reminding the reader of the definition and key properties of Lyndon words.

Let k be a positive integer, $X := \{x_1 < x_2 < \cdots < x_k\}$ be a totally ordered alphabet, and X^+ be the free monoid generated by X. The monoid X^+ is totally ordered via the lexicographic ordering. For each integer d, there is an action of the symmetric group S_d on d letters on the words of X^+ of length d; an element $\sigma \in S_d$ maps $x_{i_1} \cdots x_{i_k}$ to $x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(d)}}$. If w is a word of length d, we call a word w' of length d a rotation of w, if it is the image of w under some power of the d-cycle $(12 \cdots d)$. We denote the length of a word $w \in X^+$ by $l(w)$.

A maximum element in a set of rotations is called a *Lyndon word.* The Chen–Fox–Lyndon factorization theorem states that every word $w \in X^+$ factors uniquely as a product of Lyndon words $w = w_1 w_2 \cdots w_d$ such that $w_1 \geq w_2 \geq \cdots \geq w_d$ [CFL58]. Given a word w with Lyndon factorization $w_1 w_2 \cdots w_d$, we define the sign of w, as the product $\mathrm{sign}(w) := \prod_{i=1}^{d} (-1)^{l(w_i)-1}$.

We now state Amistur's formulae. Let G be any topological group, $D : A[G] \to A$ be a determinant of degree n, and B be an A-algebra. Let $x_1, \ldots, x_k \in B[G]$. If w is a word on $x_1, \ldots, x_k$ with Lyndon factorization $w_1^{e_1} \cdots w_d^{e_d}$ with $w_1 > w_2 \cdots > w_d$, we define

$$c(w) := \prod_{i=1}^{d} c_{e_i}(w_i) \in B.$$

For each positive integer $i \leq n$, let $\mathcal{L}_i$ be the set of words on $x_1, \ldots, x_k$ of length i. Amistur's formula states:

$$c_i(x_1 + \cdots + x_n) = \sum_{w \in \mathcal{L}_i} \mathrm{sign}(w) c(w). \tag{2.1}$$

For a proof that Amistur's formulae hold, we refer the reader to [Che14, Lemma 1.12].

Proof of Lemma 2.1.2. Let W_m be the set of words on the alphabet $G_{\mathbf{Q}} \cup \{m\}$ which contain m as a letter. Fix an A-algebra B, and let Z be the ideal of B generated by the value of the coefficients $c_i(w)$ of $P_{D,w}$ for all $w \in W$ and $i \geq 1$. Let $x' = \sum_{g\in G} b_g g \in B[G]$. By Amistur's formulae and homogeneity of the coefficients c_i, one sees that $c_i(mx')$ lies in Z. Let $w := w_1 m w_2 \in W_m$, we claim $c_i(w) = c_i(mw_2w_1)$. Assuming this, one observes that if $c_i(mx) = 0$ for all $x \in A[G]$, then $Z = 0$ and hence $m \in \ker(D)$.

That $c_i(w) = c_i(mw_2w_1)$ is a consequence of the following general fact:

Claim 1. *Let $x, y \in A[G]$ then $P_{D,xy}(X) = P_{D,yx}(X)$.*

The proof of the claim is standard from linear algebra. If $x \in A[G]$ is invertible, then:

$$P_{D,xy}(X) = D(1 + xyX) = D(x^{-1} + yX)D(x) = D(x)D(x^{-1} + yX) = P_{D,yx}(X). \tag{2.2}$$

To argue the general case, one makes the following "limiting argument." Let $A[\epsilon]$ be a polynomial ring over A. Then $x' = (1+\epsilon x)$ is invertible in $A[\epsilon][G_{\mathbf{Q}}]/\epsilon^k$ for all k, and so the argument in equation 2.2 shows:

$$D(1 + x'yt) = D(1 + yx't)$$

in $A[t,\epsilon]/\epsilon^k$ for all k. By the naturally of D, it follows $D(1 + x'yt) = D(1 + yx't)$ in $A[\epsilon, t]$. Substituting along the map $A[t, \epsilon] \to A[t, \epsilon^{\pm}]$ which sends t to $\epsilon^{-1}t$, we observe:

$$D(1 + (\epsilon + x)yt) = D(1 + y(\epsilon + x)t) \in A[t, \epsilon^{\pm}].$$

Again invoking the naturality of D, we have $D(1 + (\epsilon + x)yt) = D(1 + y(\epsilon + x)t) \in A[t, \epsilon]$. Evaluating ϵ at 0, we conclude:

$$P_{D,xy}(X) = P_{D,yx}(X).$$

□

The previous lemma is an answer to our question: an element $1 - h$ lies in the kernel of a determinant if and only if, each of the characteristic polynomial coefficients $c_i((1 - h)x)$ vanish for all $x \in A[G]$. Let H be a subgroup of G. Define $A^{H=1}$ to be the quotient of A by the closure of the ideal generated by $c_i((1 - h)x)$ for all $h \in H$, and $x \in A[G]$ and $i \in \mathbf{Z}_+$. Then the property that "H lies in the kernel of the representation" is interpolated by $A^{H=1}$. In particular, one may interpolate the property of "unramifiedness at p." Towards this end, fix, once and for all, an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$, and hence an identification of $G_{\mathbf{Q}_p}$ with a decomposition subgroup of $G_{\mathbf{Q}}$. Let I_p denote the inertia group at p. Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant and define $A^{\mathrm{un}} := A^{I_p=1}$. The algebra A^{un} interpolates the property of representations being unramified. We say a determinant is unramified if $A^{\mathrm{un}} = A$. From this discussion, one may easily deduce the following:

Lemma 2.1.3. *A determinant $D : A[G] \to A$ whose kernel contains $h - 1$ for all $h \in H \subset G$ factors through $A[G/N]$, where N is the normal closure of H in G.*

Unramifedness for determinants is not a local property, i.e. it is not the case that one can determine if a determinant is unramified given only the restriction of that determinant to a decomposition group at p. This is true even for determinants valued in algebraically closed fields; if V is a finite dimensional, irreducible representation defined over an algebrically closed field, the list of characteristic polynomials for each element in $G_{\mathbf{Q}_p}$ is in general not sufficient to determine if that representation is unramified. What can be determined from this list of characteristic polynomials is if the semi-simplification of the local representation is unramified, i.e. if there is a complete $G_{\mathbf{Q}_p}$-stable flag on V such that the associated graded representation is unramified.

Classically, a p-adic representation $V/\overline{\mathbf{Q}}_p$ of $G_{\mathbf{Q}}$ is called ordinary if V admits a complete $G_{\mathbf{Q}_p}$-stable flag such the restriction of the associated graded representation to I_p is a monotonically non-decreasing sequence of powers of the cyclotomic character. The sequence of powers $(w_1, \ldots, w_n) \in \mathbf{Z}^n$ of the cyclotomic character by which I_p acts is called the weight. Those representations that are ordinary of weight $(0, \ldots, 0)$ are exactly those whose local representation at p have an unramified semi-simplification.

In the next section, we show that the property that a representation is "ordinary of weight $\mathbf{w} := (w_1, \ldots, w_n)$" is interpolatable by a closed subscheme $\mathrm{Spec}(A^{\mathrm{ord},\mathbf{w}}) \subset \mathrm{Spec}(A)$. In the case $\mathbf{w} = (0, \ldots, 0)$, we show $\mathrm{Spec}(A^{\mathrm{ord},\mathbf{w}}) \supseteq \mathrm{Spec}(A^{\mathrm{un}})$. In the following section, we will give a scheme theoretic criterion to check if a ordinary determinant of weight $(0, \ldots, 0)$ is unramified at p. Our proof that determinants attached to weight one Hecke algebras are unramified at p will proceed by first showing that such determinants are ordinary of weight $(0, \ldots, 0)$, and then applying this criterion.

2.2. **Ordinary Determinants.** Let V be an n-dimensional p-adic representation of $G_{\mathbf{Q}_p}$. Recall that V is said to be ordinary if there exists a sequence of non-decreasing integers $(w_1, \ldots, w_n)$ and a $G_{\mathbf{Q}_p}$-stable filtration of V :

$$0 = F_{n+1}V \subseteq F_{n-1}V \subseteq F_2V \subseteq \ldots \subseteq F_1V = V,$$

such that the graded subquotients $F_iV/F_{i+1}V$ are one dimensional and the inertia group I_p acts on $F_iV/F_{i+1}V$ by the w_i-th power of the cyclotomic character. We call such a filtration an ordinary flag on V (of weight $(w_1, \ldots, w_n)$). A p-adic representation of $G_{\mathbf{Q}}$ is called ordinary if its local representation at p is ordinary. In this section, we give a condition on the family of characteristic polynomials associated to a semi-simple $G_{\mathbf{Q}}$-representation which guarantee that that representation is ordinary. In other words, we define what it means for a determinant to be ordinary.

The notion of ordinariness for representations depends on an auxiliary piece of information: the existence of an ordinary flag. If one wishes to express which relations occur in a family of characteristic polynomials associated to an ordinary representation, one should preliminarily do so with a fixed choice of ordinary flag. Let be $F_\bullet$ be an ordinary flag on V. From $F_\bullet$ one obtains an *ordered* sequence of characters $(\chi_1, \ldots \chi_n)$ of $G_{\mathbf{Q}_p}$; the action of $G_{\mathbf{Q}_p}$ on the graded quotient $F_iV/F_{i+1}V$ being through χ_i. The collection of characters $(\chi_1, \ldots \chi_n)$ determines the characteristic polynomials of every element $g \in G_{\mathbf{Q}_p}$. Specifically,

$$P_{D,g}(X) = \prod_{i=1}^{d}(1 - \chi_i(g)X).$$

However, this relation only depends on the semi-simplification of $V|_{G_{\mathbf{Q}_p}}$, and therefore does not depend on the choice of ordinary flag.

To see relations imposed by the choice of flag $F_\bullet$ one must consider the characteristic polynomials of elements in $\overline{\mathbf{Q}}_p[G_{\mathbf{Q}}]$ outside $\overline{\mathbf{Q}}_p[G_{\mathbf{Q}_p}]$. Observe for each i and any sequence of elements $g_1, \ldots, g_n \in G_{\mathbf{Q}_p}$, the image

$$(g_i - \chi_i(g_i)) \cdots (g_1 - \chi_1(g_1))V \subset F_iV.$$

Since, F_iV is an $n+1-i$ dimensional subspace of the n dimensional space V, given any operator $m \in \mathrm{End}(V)$, the operator

$$(g_i - \chi_i(g_i)) \cdots (g_1 - \chi_1(g_1))m$$

has 0 as an eigenvalue with multiplicity at least i. It follows that the trailing i coefficients of the characteristic polynomial of

$$(g_i - \chi_i(g_i)) \cdots (g_1 - \chi_1(g_1))m$$

must vanish for all m in the group ring of $G_{\mathbf{Q}}$.

We record these conditions as our interpolation of ordinariness. Let $\epsilon : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ be the cyclotomic character and W_p be the Weil group of $\mathbf{Q}_p$. Our notion of ordinariness for determinants begins, like the analogous notion for representations, with the existence of certain additional structures. For representations, it is the existence of a weight and filtration. For a determinant it is:

Definition 2.2.1. *Let B be a $\mathbf{Z}_p$-algebra. Let n be a positive integer, and $\mathbf{w} := (w_1, \dots w_n)$ be a sequence of non-decreasing integers. A B-valued, ordinary determinant of degree n, weight $\mathbf{w}$, and local type $(\chi_1, \dots \chi_n)$ is a pair $(D, (\chi_1, \dots \chi_n))$ consisting of:*

1. *a continuous B-valued determinant $D : B[G_{\mathbf{Q}}] \to B$ of degree n,*
2. *an ordered collection $(\chi_1, \dots \chi_n)$ of characters $\chi_i : W_p \to B^\times$,*

such that:

1. $\chi_i|_{I_p} = \epsilon^{w_i}$,
2. *for all $g \in W_p$ the characteristic polynomial of g is:*
$$P_{D,g}(X) = \prod_{i=1}^{n} (1 - \chi_i(g)X),$$
3. *for each positive integer $i \leq n$, sequence of elements $g_1 \dots g_i \in W_p$, and $x \in B[G_{\mathbf{Q}}]$:*
$$c_j((g_i - \chi_i(g_i))(g_{i-1} - \chi_{i-1}(g_{i-1})) \cdots (g_1 - \chi_1(g_1))x) = 0,$$
if $j > n - i$.

We say a determinant is ordinary if those additional structures exist over a faithful extension of scalars. That is:

Definition 2.2.2. *Let A be a $\mathbf{Z}_p$-algebra. Let d be a positive integer and $\mathbf{w} := (w_1, \dots w_n)$ be a sequence of non-decreasing integers. An A-valued determinant $D : A[G_{\mathbf{Q}}] \to A$ of degree d is called called ordinary of weight $\mathbf{w}$, if there exists a faithful A-algebra B and an ordered collection $(\chi_1, \dots \chi_n)$ of characters $\chi_i : W_p \to B^\times$, such that*

$$(D : B[G_{\mathbf{Q}}] \to B, (\chi_1, \dots \chi_n))$$

is an ordinary determinant. We say that B realizes an ordinary filtration for D of weight $(w_1, \dots w_n)$ and local type $(\chi_1, \dots \chi_n)$.

Like the notion of unramifiedness for determinants, our definition of ordinariness for determinants is *not* local, i.e. does not solely depend on the determinant restricted to a decomposition group at p.

We note that the notion of ordinariness with a local type is functorial. If $D : B[G_{\mathbf{Q}}] \to B$ is a determinant of weight $\mathbf{w}$ with local type $(\chi_1, \dots \chi_n)$ and $\phi : B \to B'$ is a ring map, then D_B is ordinary with local type $(\phi \circ \chi_1, \dots \phi \circ \chi_n)$. The naked notion of ordinariness for determinants does *not* necessarily persist under base change. It is therefore easier in practice to work with the former notion.

In the discussion above, we saw that every ordinary representation gives rise to an ordinary determinant of the same weight. In the next theorem, we show the converse is true.

Theorem 2.2.3. *Let* $\mathbf{w} := (w_1, \dots w_n)$ *be a non-decreasing sequence of integers. The map which associates a determinant to a representation induces a bijection from the set of isomorphism classes of weight* $\mathbf{w}$ *ordinary, semi-simple* n*-dimensional representations of* $G_{\mathbf{Q}}$ *to the set ordinary determinants* $D : \overline{\mathbf{Q}}_p[G_{\mathbf{Q}}] \to \overline{\mathbf{Q}}_p$ *of degree* n *and weight* $\mathbf{w}$*.*

Proof. By [Che14, Theorem 2.12], one knows that the map which associates a determinant to a representation induces a bijection from the set of isomorphism classes of n-dimensional, semi-simple representations of $G_{\mathbf{Q}}$ over $\overline{\mathbf{Q}}_p$ to the set of degree n, determinants $D : \overline{\mathbf{Q}}_p[G_{\mathbf{Q}}] \to \overline{\mathbf{Q}}_p$. What remains to be shown is that if a determinant associated to a representation is ordinary of weight $\mathbf{w}$, then that representation is ordinary of weight $\mathbf{w}$. As the weight of an ordinary representation is an invariant of the local determinant at p (which by assumptions (1) and (2) in the definition of ordinariness necessarily has weight $\mathbf{w}$), it suffices to show that V is ordinary of *some* weight.

Let V be a n-dimensional, semi-simple representation of $G_{\mathbf{Q}}$, whose associated determinant D is ordinary. By definition, there exists a faithful $\overline{\mathbf{Q}}_p$-algebra B that realizes an ordinary filtration for D. Let $\mathbf{w} := (w_1, \dots w_n)$ be a weight and $(\chi_1, \dots \chi_n)$ be a local type for this extension. Let $B^{\min}$ be the A-algebra generated by the values of $\chi_1(g), \dots, \chi_n(g)$ as g ranges over the elements $g \in W_p$. Then $B^{\min}$ realizes an ordinary filtration for D of local type $(\chi_1, \dots \chi_n)$. As the image of I_p under χ_i lies in $\mathbf{Z}_p$, the ring $B^{\min}$ is generated as an algebra by the images of any Frobenius element at p under $\chi_1, \dots, \chi_n$. In particular, $B^{\min}$ is a finite type A-algebra. It follows that there is a ring theoretic section of the algebra map $\overline{\mathbf{Q}}_p \hookrightarrow B^{\min}$. By post-composing the characters $(\chi_1, \dots \chi_n)$ with such a section, we observe that $\overline{\mathbf{Q}}_p$ itself realizes an ordinary filtration for D. Hence, we may assume $B = \overline{\mathbf{Q}}_p$.

We claim each of the characters $\chi_i : W_p \to \overline{\mathbf{Q}}_p^\times$ extend to a continuous character of $G_{\mathbf{Q}_p}$. Consider the set P consisting the of characteristic polynomials $P_{D,g}(X)$ where $g \in G_{\mathbf{Q}_p}$. As D is continuous, the set of valuations of the coefficients of polynomials in P is finite. Hence, the set Val of valuations of roots of polynomials in P is finite. For each i and each of the elements $g \in W_p$, the value $\chi_i(g)^{-1}$ is a root of $P_{D,g}(X)$, and therefore the valuation $\chi_i(g)$ lies in Val. On the other hand, the image of $\chi_i(W_p)$ under the valuation map is a subgroup of $\mathbf{Q}$. We conclude $|\chi_i(g)| = 1$ for all $g \in W_p$. This implies χ_i extends to a continuous character of $G_{\mathbf{Q}_p}$. We denote the unique extension of $\chi_i : W_p \to \overline{\mathbf{Q}}_p^\times$ to $G_{\mathbf{Q}_p}$ by the same symbol.

Finally, we show V admits an ordinary flag. For each sequence of elements $g_{n-1}, \dots, g_1 \in W_p$, consider the subspace:

$$V(g_{n-1}, \dots, g_1) := (g_{n-1} - \chi_{n-1}(g_{n-1})) \cdots (g_1 - \chi(g_1))V.$$

By assumption, the characteristic polynomial of

$$(g_n - \chi_n(g_n))(g_{n-1} - \chi_{n-1}(g_{n-1})) \cdots (g_1 - \chi(g_1))x$$

is identically 1 for all $x \in \overline{\mathbf{Q}}_p[G_{\mathbf{Q}}]$. As V is semi-simple, it follows the element

$$(g_n - \chi_n(g_n))(g_{n-1} - \chi_{n-1}(g_{n-1})) \cdots (g_1 - \chi(g_1))$$

acts on V by 0, and so $V(g_{n-1}, \dots, g_1)$ is contained in the χ_n isotopic subspace $V^{\chi_n} \subseteq V$. Consequently, the quotient V/V^{χ_n} is annihilated by $(g_{n-1} - \chi_{n-1}(g_{n-1})) \cdots (g_1 - \chi(g_1))$.

Proceeding inductively one constructs a filtration of V whose graded pieces are isomorphic to $(\overline{\mathbf{Q}}_p(\chi_i))^{k_i}$ for some sequence of exponents k_i. As the weights of the characters $\chi_1, \dots \chi_n$ are increasing, this filtration refines to an ordinary flag for V, and hence V is ordinary. □

Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant of degree n and $\mathbf{w} = (w_1, \dots, w_n)$ be a nondecreasing sequence of integers. Our previous theorem shows that any ordinary $\overline{\mathbf{Q}}_p$-point of $\mathrm{Spec}(A)$ arises from an ordinary representation. Next we show that there is a unique, maximal, closed subscheme of $\mathrm{Spec}(A)$ over which D is ordinary of weight $\mathbf{w}$. We call this subscheme the ordinary locus of $\mathrm{Spec}(A)$. If there is no A-algebra A' such that $D_{A'}$ is ordinary of weight $\mathbf{w}$, this locus is empty. Therefore, we assume there exists such an A-algebra A', and construct a quotient $A^{\mathrm{ord},\mathbf{w}}$ of A, which is the initial A-algebra such that the base change of D to $A^{\mathrm{ord},\mathbf{w}}$ is ordinary of weight $\mathbf{w}$.

Given a determinant D', to exhibit that D' is ordinary, one must find a faithful extension of scalars which realizes an ordinary filtration. Specifically, one must find a local type, i.e. a sequence of characters $(\chi_1, \dots, \chi_n)$ of W_p valued in the scalar extension, which satisfy certain compatibilities with that determinant. In the case that such a sequence exists, the restriction of $(\chi_1, \dots, \chi_n)$ to the inertia group at p is determined by the weight. Hence, one may specify the characters uniquely by specifying the values $\chi_1(\phi), \dots, \chi_n(\phi)$ for some choice of Frobenius element ϕ at p. Fix an element $\phi \in G_{\mathbf{Q}}$ such that ϕ is a Frobenius element at p and $\epsilon(\phi) = 1$. Since D' is ordinary, the characteristic polynomial $P_{D',\phi}(X)$ of ϕ is equal to $\prod_{i=1}^n (1 - \chi_i(\phi)X)$. To construct $A^{\mathrm{ord},\mathbf{w}}$, we will consider the finite, flat cover $\mathrm{Spec}(A_\phi)$ of $\mathrm{Spec}(A)$, which parameterizes *complete, ordered sets of zeros* $(r_1, \dots, r_n)$ for the classically normalized characteristic polynomial $X^n P_{D',\phi}(X^{-1})$ of ϕ. For each i, there is a unique unramified character $\chi(r_i)$ of $G_{\mathbf{Q}_p}$ valued in A_ϕ, which sends ϕ to r_i. Given such characters, one may cut out from $\mathrm{Spec}(A_\phi)$ a maximal, closed subscheme $\mathrm{Spec}(A_\phi^{\mathrm{ord},\mathbf{w}})$ over which D is ordinary of weight $\mathbf{w}$, and an ordinary filtration is realized with local type $(\chi(r_1)\epsilon^{w_1}, \dots, \chi(r_n)\epsilon^{w_n})$. We conclude the construction by showing that the scheme-theoretic image of $\mathrm{Spec}(A_\phi^{\mathrm{ord},\mathbf{w}})$ in $\mathrm{Spec}(A)$ is the ordinary locus.

We now construct $A^{\mathrm{ord},\mathbf{w}}$ in earnest. The characteristic polynomial of ϕ under D equals:

$$P_{D,\phi}(X) = 1 + \sum_{i=1}^{n} (-1)^i c_i(\phi) X^i,$$

where each coefficient $c_i(\phi) \in A$. Let $e_i(r_1, \dots r_n)$ be the i-th symmetric function of degree d in the indeterminants $r_1, \dots, r_n$. Explicitly,

$$e_i(r_1, \dots r_n) := \sum_{\substack{S \subset n \\ |S| = i}} \prod_{i \in S} r_i \in A[r_1, \dots, r_n].$$

Define

$$A_\phi := A[r_1, \dots, r_n]/(c_i(\phi) - e_i(r_1, \dots r_n) : 1 \leq i \leq n).$$

By the fundamental theorem of symmetric polynomials, the ring A_ϕ is a free A-algebra of degree $n!$. The monomials $r_1^{e_1} \cdots r_n^{e_n}$ with $0 \leq e_i < i$ constitute an A-basis for A_ϕ.

For $g \in W_p$, we denote by $|g| \in \mathbf{Z}$ the unique power such that $g \equiv \phi^{|g|} \mod I_p$. Define $A_\phi^{\mathrm{ord},\mathbf{w}}$ to be the quotient of A_ϕ obtained by imposing that for all $g \in W_p$ and $i \in \{1, \dots n\}$,

$$c_i(g) = e_i(r_1^{|g|}\epsilon(g)^{w_1}, \dots, r_n^{|g|}\epsilon(g)^{w_n}), \tag{2.3}$$

and for each $i \le n$ and every sequence of elements $g_1, \dots, g_i \in W_p$ and element $x \in A_\phi[G_{\mathbf{Q}}]$, the coefficients

$$c_j((g_i - r_i^{|g_i|}\epsilon(g_i)^{w_i})(g_i - r_i^{|g_{i-1}|}\epsilon(g_{i-1})^{w_{i-1}}) \cdots (g_1 - r_1^{|g_1|}\epsilon(g_1)^{w_1})x) = 0 \tag{2.4}$$

for all $j > n - i$. Define $A^{\mathrm{ord},\mathbf{w}}$ to be the image of A in $A_\phi^{\mathrm{ord},\mathbf{w}}$.

One observes the following:

Essentially by construction, the base change of D to $A^{\mathrm{ord},\mathbf{w}}$ is ordinary of weight $\mathbf{w}$. The image of each of the indeterminants r_i in A_ϕ is invertible. If $\chi(r_i) : W_p \to (A_\phi^{\mathrm{ord},\mathbf{w}})^\times$ is the unramified character which maps ϕ to r_i, one observes that the base change of D to $A_\phi^{\mathrm{ord},\mathbf{w}}$ is ordinary of weight $\mathbf{w}$ with local type $(\chi(r_1)\epsilon^{w_1}, \dots, \chi(r_n)\epsilon^{w_n})$. Since, $A^{\mathrm{ord},\mathbf{w}}$ injects into $A_\phi^{\mathrm{ord},\mathbf{w}}$ the base change of D to the A-algebra $A^{\mathrm{ord},\mathbf{w}}$ is ordinary.

The algebra $A_\phi^{\mathrm{ord},\mathbf{w}}$ represents the functor on A-algebras of "local types of weight $\mathbf{w}$*".* If B is any A-algebra which is ordinary of weight $\mathbf{w}$ and local type $\chi := (\chi_1, \dots, \chi_n)$, then $(\chi_1(\phi), \dots, \chi_n(\phi))$ is a complete set of roots for $P_{D,\phi}$. The resulting A-algebra map $f_\chi : A_\phi \to B$ sends r_i to $\chi(r_i)$. This map necessarily factors through $A_\phi^{\mathrm{ord},\mathbf{w}}$. Since the characters $(\chi_1, \dots, \chi_n)$ are determined by their weight and value on ϕ, the map f_χ determines χ. It follows that the set of B-valued local types of weight $\mathbf{w}$ is in natural bijection with $\mathrm{Hom}_A(A_\phi^{\mathrm{ord},\mathbf{w}}, B)$. Given this functorial description, we see $\mathcal{A}_\phi^{\mathrm{ord},\mathbf{w}} = \mathcal{A} \otimes_A \mathcal{A}^{\mathrm{ord},\mathbf{w}}$ for any A-algebra $\mathcal{A}$.

The assignment $A \rightsquigarrow A^{\mathrm{ord},\mathbf{w}}$ is an interpolation of the notion of ordinariness of weight $\mathbf{w}$. Any subalgebra of an $A_\phi^{\mathrm{ord},\mathbf{w}}$-algebra is necessarily ordinary. As $A_\phi^{\mathrm{ord},\mathbf{w}}$ is finite over $A^{\mathrm{ord},\mathbf{w}}$, every geometric point $\pi : A^{\mathrm{ord},\mathbf{w}} \to F$ lifts to a geometric point of $A_\phi^{\mathrm{ord},\mathbf{w}}$, and hence is ordinary of weight $\mathbf{w}$. Hence, $\mathrm{Spec}(A^{\mathrm{ord},\mathbf{w}})$ contains all ordinary points of A.

Conversely, if $\mathcal{A}$ is an ordinary A-algebra of weight $\mathbf{w}$, then since $\mathcal{A} \otimes_A A_\phi^{\mathrm{ord},\mathbf{w}} = \mathcal{A}_\phi^{\mathrm{ord},\mathbf{w}}$ the map $A \to \mathcal{A}^{\mathrm{ord},\mathbf{w}}$ factors through $\mathcal{A}^{\mathrm{ord},\mathbf{w}}$. It follows that the assignment $A \rightsquigarrow A^{\mathrm{ord},\mathbf{w}}$ is functorial and every geometric point of $A^{\mathrm{ord},\mathbf{w}}$ is ordinary.

We record these observations in the following theorem:

Theorem 2.2.4. *Let A be a $\mathbf{Z}_p$-algebra and $D : A[G_{\mathbf{Q}}] \to A$ be a determinant of degree n. Let $\mathbf{w} = (w_1, \dots, w_n)$ be nondecreasing sequence of integers.*

1. *The base change of D to $A^{\mathrm{ord},\mathbf{w}}$ is ordinary of weight* $\mathbf{w}$.
2. *The determinant D is ordinary if and only if $A^{\mathrm{ord},\mathbf{w}}$ equals A.*
3. *If A' is an A-algebra such that the base change of D to A' is ordinary of weight* $\mathbf{w}$*, then the structure map $st : A \to A'$ factors through $A^{\mathrm{ord},\mathbf{w}}$.*
4. *The algebra $A_\phi^{\mathrm{ord},\mathbf{w}}$ represents the set valued functor on A-algebras that assigns to an A-algebra B the set of B-valued local types for D_B of weight* $\mathbf{w}$.
5. *If B is any $A_\phi^{\mathrm{ord},\mathbf{w}}$-algebra and $A' \subseteq B$ is an A-subalgebra, then the base change of D to A' is ordinary of weight* $\mathbf{w}$.
6. *The base change of D to any geometric point of $A^{\mathrm{ord},\mathbf{w}}$ is ordinary of weight* $\mathbf{w}$.

We remark that if A is a $\mathbf{Z}_p$-algebra in which $p^n = 0$, and $\mathbf{w_1}, \mathbf{w_2} \in \mathbf{Z}^n$ are two weights that are congruent modulo $(p-1)p^{n-1}$, then $A^{\mathrm{ord},\mathbf{w_1}} = A^{\mathrm{ord},\mathbf{w_2}}$. Hence, given any $\mathbf{Z}_p$-algebra A, a determinant $D : A[G_{\mathbf{Q}}] \to A$, and a weight $\mathbf{w_1}$, the quotient $A^{\mathrm{ord},\mathbf{w_1}}$ is a p-adic limit of ordinary quotients of arbitrary large integer weights. We will use this observation in section 3 to show that the determinant attach an ordinary determinant valued in the *ordinary* p-adic Hecke algebra of weight one.

In order for a p-adic representation of $G_{\mathbf{Q}}$ to be unramified at p, it must be ordinary of weight $(0,\dots,0)$. Our next theorem shows the same holds for our interpolations of these two properties to determinants. In the next section, we will give a scheme theoretic criterion to determine if a determinant that is ordinary of weight $(0,\dots,0)$ is unramified.

Theorem 2.2.5. *Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant which is unramified at p. Then D is ordinary of weight $\mathbf{0} := (0,\dots,0)$. If $D : A[G_{\mathbf{Q}}] \to A$ is any determinant, then the ordinary weight $\mathbf{0}$ locus $\mathrm{Spec}(A^{\mathrm{ord},\mathbf{0}})$ contains the unramified locus $\mathrm{Spec}(A^{\mathrm{un}})$.*

Proof. The second statement follows immediately from the first. We show the first holds.

Assume $D : A[G_{\mathbf{Q}}] \to A$ is unramified at p. Let $\chi(r_i) : W_p \to A_\phi$ be the unramified character mapping ϕ to r_i. We claim A_ϕ realizes an ordinary filtration for D of weight $\mathbf{0}$ and local type $(\chi(r_1),\dots,\chi(r_n))$.

Since D is unramified at p, the kernel of D contains $g-1$ for all $g \in I_p$. This implies, the characteristic polynomial $P_{g,D}(X)$ for $g \in W_p$ and the coefficient

$$c_j((g_i - \chi(r_i) \cdots (g_1 - \chi(r_1))m)$$

for all $i, j > 0$ and $g_1,\dots,g_n \in W_p$ and $m \in A[g_{\mathbf{Q}}]$ only depends on $g, g_1, \dots, g_n \mod I_p$. Hence, to show D is ordinary it suffices to show

$$P_{\phi^k,D}(X) = \prod (1 - r_i^k X) \tag{2.5}$$

for all $k \in \mathbf{Z}$, and

$$c_j((\phi^{k_i} - r_i^{k_i}) \cdots (\phi^{k_1} - r_1^{k_1})m) = 0 \tag{2.6}$$

for all $k_1,\dots,k_i \in \mathbf{Z}$, elements $m \in A[G_{\mathbf{Q}}]$ and $j \geq n-i$.

There is a certain ideal $\mathrm{Rel}_D \subset A$, which we now define, that vanishes if and only if relations 2.5 and 2.6 hold for D. The algebra A_ϕ is free over A of rank $n!$ on the basis $\mathcal{B} := (r_1^{k_1} \cdots r_n^{k_n} : 0 \leq k_i < i)$. Using Amistur's formula, one may expand

$$c_j((\phi^{k_i} - r_i^{k_i}) \cdots (\phi^{k_1} - r_1^{k_1})m) \in A_\phi$$

and write it as an A-linear expression on the basis $\mathcal{B}$. For each $r \in \mathcal{B}$, let $c_r(j, k_1, \dots, k_i, m) \in A$ be the coefficient of r in this expansion. Let Rel_D be the ideal of A generated by

$$c_i(j, k_1, \dots, k_i, m) \text{ and } c_i(\phi^k) - e_i(r_1^k, \dots, r_n^k)$$

for all sequences $k, k_1, \dots k_i \in \mathbf{Z}$, integers $i > 0$ and $j > n-i$. The ideal $\mathrm{Rel}_D = 0$ in A if and only if the proposition holds for D. We prove the theorem by showing Rel_D vanishes.

We begin by considering the case in which $F := A$ is an algebraically closed field. In this case, D arises from a genuine representation, and one can argue using matrices. Furthermore, to show $\mathrm{Rel}_D = 0$, it suffices to do so in any of the specializations of $F_\phi \to F$, i.e. when $r_1, \dots r_n$ are genuine eigenvalues of ϕ. Given a genuine representation, one may put ϕ into a Jordan normal form in which the specializations of $r_1, \dots, r_n$ appear in that order on the diagonal of ϕ. In particular, this representation is ordinary with local type $(\chi(r_1), \dots, \chi(r_n))$. It follows that 2.5 and 2.6 hold and $\mathrm{Rel}_D = 0$ in F.

Given a general determinant $D : A[G_{\mathbf{Q}}] \to A$ which is unramified at p, one may hope to lift D to a determinant $\widehat{D}$ which is unramified at p and is valued in some domain $\widehat{A} \to A$.

Assuming this, one would deduce from the previous paragraph that $\mathrm{Rel}_{\widehat{D}}$ vanishes in the algebraic closure of the fraction field of $\widehat{A}$, and hence that it vanishes in $\widehat{A}$. From this one would deduce Rel_D vanishes in A and prove the theorem. This however will be impossible, as relations in $G_{\mathbf{Q}}$ may prevent lifts from existing.

Nonetheless, we will deduce the general case from the case of a determinant valued an algebraically closed field. We will use an idea of Francesco Vaccarino: any determinant lifts to a determinant valued in an algebraically closed field after *enough* relations have been removed *from the group.*

In fact, Vaccarino will remove so many relations that our group will no longer be a group. Let G be any group and X_G be the free non-commutative monoid generated by the symbols x_g for $g \in G$. Denote the free non-commutative algebra on X_G by $\mathbf{Z}\{X_G\}$, and let $F_n(X_G)$ be the polynomial ring over $\mathbf{Z}$ generated by symbols $x(g)_{i,j}$ for $g \in G$ and $i, j \in \{1, \ldots, n\}$. Let $\rho : \mathbf{Z}\{X_G\} \to M_n(F_n(X_G))$ be the n-dimensional representation which maps $x \mapsto [x(g)_{i,j}]_{i,j}$ and $E_n(X_G)$ be the subring $F_n(X_G)$ generated by characteristic polynomial coefficients of $\rho(w)$ for all $w \in \mathbf{Z}[X_G]$. For any ring A, let $\pi_A : \mathbf{Z}\{X_G\} \to A[G]$ be the ring homorphism sending $x_g \to g$. Vaccarino shows:

Theorem 2.2.6 (Vaccarino [Vac08, Theorem 6.1], also see [Che14, pg. 15, Theorem 1.15])**.** *Let $D : A[G] \to A$ be a determinant. Then there exists a unique ring homomorphism $\phi_D : E_d(X_G) \to A$ such that $\phi_D \circ \det(\rho) = D \circ \pi_A$.*

We conclude our proof using Vaccarino's result. In analogy to A_ϕ, one defines an $E_n(X_G)$-algebra $E_n(X_G)_{x_\phi}$ by quoting the polynomial ring $E_n(X_G)[r_1, \ldots, r_n]$ by the coefficients of the polynomial

$$\rho(1 - x_g X) - \prod_{i=1}^{n}(1 - r_i X).$$

Then replacing A_ϕ and $E_n(X_G)_{x_\phi}$ in the construction of Rel_D one creates an ideal

$$\mathrm{Rel}_{\det(\rho)} \subseteq E_n(X_G).$$

The specialization $\phi_D(Rel_{\det(\rho)}) = Rel_D$. The argument to that $\mathrm{Rel}_D = 0$ when D is valued in an algebraically closed field, generalizes to shows $\mathrm{Rel}_{\det(\rho)}$ vanishes in the fraction field of the algebraic closure of $F_d(X_G)$. We conclude Rel_D vanishes in A. □

Thus, a determinant which is unramified at p is ordinary of weight $\mathbf{0}$. The converse is not true. In the next section, we will consider the question of describing the unramifed locus inside the ordinary weight $\mathbf{0}$ locus, and give a criterion to determine if a degree n determinant is unramified.

2.3. **A Criterion for Unramifiedness.** To motivate our criterion for unramifiedness, consider the problem of determining if an n-dimensional, p-distinguished[4] $G_{\mathbf{Q}}$-representation $V/\overline{\mathbf{Q}}_p$ is unramified at p. For such a representation, each distinct ordinary flag on V gives rise to a distinct complete ordering of the n eigenvalues of ϕ. The representation V is unramified if and only if every ordering of the eigenvalues of ϕ can be realized by an ordinary flag. Hence, a p-distinguished representation is unramfied if and only if it admits at least $n!$ distinct ordinary flags.

[4]An n-dimensional $G_{\mathbf{Q}}$-representation V defined over an algebraically closed field $\overline{F}$ is called p-distinguished if ϕ acts with n-distinct eigenvalues

For a determinant $D : A[G_{\mathbf{Q}}] \to A$, the scheme $\mathrm{Spec}(A_\phi^{\mathrm{ord},\mathbf{0}})$ parameterizes complete, ordered sets of zeros of the characteristic polynomial $P_{D,\phi}(X)$ which interpolate between those orderings that arise from ordinary flags of weight $\mathbf{0}$. If x is a p-distinguished geometric point of $\mathrm{Spec}(A)$, then the condition that V_x admits at least $n!$ distinct ordinary flags is equivalent to the fiber of $\mathrm{Spec}(A_\phi^{\mathrm{ord},\mathbf{0}})$ above x being at least $n!$-dimensional. We show that this numerical criterion for unramifiedness generalizes to all determinants.

Theorem 2.3.1 (Numerical Criterion for Unramifiedness)**.** *A determinant $D : A[G_{\mathbf{Q}}] \to A$ of degree n is unramified if and only if it is ordinary of weight $\mathbf{0}$ and the A-algebra $A_\phi^{\mathrm{ord},\mathbf{0}}$ admits an A-module quotient which is free over A of finite rank equal to $n!$. In this case, we will have that $A_\phi \simeq A_\phi^{\mathrm{ord},\mathbf{0}}$ is free of rank $n!$.*

As suggested in Remark 3.22 of [CG], the main reason why one might expect this theorem to be true is that an ordinary representation should be unramified if and only if it is ordinary for every possible ordering of the eigenvalues of Frobenius.

We make the following definition:

Definition 2.3.2. *Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant of degree n. Let $\mathbf{0} := (0, \ldots 0)$. If B is an A-algebra, define $B_\phi := B \otimes_A A_\phi$ and $B_\phi^{\mathrm{ord},\mathbf{0}} := B \otimes_A A_\phi^{\mathrm{ord},\mathbf{0}}$. We say the determinant D_B is split of weight $\mathbf{0}$ if the induced map $B_\phi \to B_\phi^{\mathrm{ord},\mathbf{w}}$ is an isomorphism.*

We remark that for an A-algebra B, the rings B_ϕ and $B_\phi^{\mathrm{ord},\mathbf{w}}$ are naturally isomorphic to those of the same names attached to the determinant D_B.

We claim that the base change of D to an arbitrary A-algebra is split of weight $\mathbf{0}$ if and only if it is unramified at p. Since there is a universal unramified A-algebra, such an equivalence would dictate that there must exist a universal A-algebra which is split of weight $\mathbf{0}$. This is the first theorem of this section.

Theorem 2.3.3. *Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant of degree n. There exists an A-algebra A^{split} such that, given an A-algebra B, the determinant D_B is split weight $\mathbf{0}$ if any only if, the structure map $A \to B$ factors through A^{split}.*

Proof. The A-algebra A_ϕ is a free the A-basis $\mathcal{B} := (r_1^{e_1} \cdots r_n^{e_n} : 0 \le e_i < i)$. For each multi-index $I = (e_1, \ldots e_n)$ with $0 \le e_i < i$, let $\pi_I : A_\phi \to A$ be the projection operator which maps an element $x \in A_\phi$ to the coefficient of $r_1^{e_1} \cdots r_n^{e_n}$ in the unique A-linear expansion of x in terms of $\mathcal{B}$. Define A^{split} to be the quotient of A by the ideal generated by the image of the kernel $\ker(A_\phi \to A_\phi^{\mathrm{ord},\mathbf{0}})$ under the various projection maps π_I, as I runs over the set of multi-indices $I = (e_1, \ldots e_n)$ with $0 \le e_i < i$. An A-algebra B is split of weight $\mathbf{0}$ if and only if the image of $B \otimes_A \ker(A_\phi \to A_\phi^{\mathrm{ord},\mathbf{0}})$ in $B_\phi = B \otimes_A A_\phi$ is the zero ideal. As $B_\phi = B \otimes_A A_\phi$ is a free B-algebra on $1 \otimes \mathcal{B}$, it follows B is split of weight $\mathbf{0}$ if and only if the structure map $A \to B$ factors through A^{split}. □

By Theorem 2.2.5, if a determinant $D : A[G_{\mathbf{Q}}] \to A$ is unramified at p, then it is ordinary of weight $\mathbf{0}$. To prove this, we showed that an ordinary filtration was realized over A_ϕ. Hence, the natural map $A_\phi \to A_\phi^{\mathrm{ord},\mathbf{0}}$ is an isomorphism. In the parlance of this section: determinants that are unramified at p are split of weight $\mathbf{0}$. Applying this to the universal case, it follows that there is a natural surjective map $A^{\mathrm{split}} \twoheadrightarrow A^{\mathrm{un}}$. Our next theorem shows that this map is an isomorphism. Consequently, an A-valued determinant is split weight $\mathbf{0}$ if and only if it is unramified at p.

Theorem 2.3.4. *Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant of degree n, then the natural map $A^{\text{split}} \to A^{\text{un}}$ is an isomorphism.*

Proof. To show that the natural map $A^{\text{split}} \twoheadrightarrow A^{\text{un}}$ is an isomorphism it suffices to show that the base change of D to A^{split} is unramified at p, i.e. that the element $g-1 \in \ker(D_{A^{\text{split}}})$ for all $g \in I_p$. As the statement of this theorem and its proof only depend on the base change of D to A^{split}, we will assume without loss of generality that D is split, i.e. that $A = A^{\text{split}}$.

By Lemma 2.1.2, an element $x \in A[G_{\mathbf{Q}}]$ lies in the kernel of the determinant D if and only if for all positive integers $i \leq n$ and all $m \in A[G_{\mathbf{Q}}]$, the characteristic polynomial coefficient $c_i(xm)$ vanishes. For each $i \leq n$, define $Z_i \subset A[G_{\mathbf{Q}}]$ to be the subset:

$$Z_i := \{x \in A[G_{\mathbf{Q}}] : c_j(xm) = 0 \text{ for all } j \leq i \text{ and } m \in A[G_{\mathbf{Q}}]\}.$$

The set Z_i is a two sided ideal of $A[G_{\mathbf{Q}}]$: it being additively closed is a consequence of Amistur's formula (formula 2.1), and it being closed under multiplication by arbitrary elements of $A[G_{\mathbf{Q}}]$ is a consequence of the identity $P_{D,mx}(X) = P_{D,xm}(X)$, as claimed and shown in the proof of Theorem 2.1.2. The ideal $Z_n = \ker(D)$. Define $Z_0 = A[G_{\mathbf{Q}}]$. To prove the proposition, we will show inductively that $g-1 \in Z_i$ for each i.

Let $i < n$ and $g \in I_p$. Assume $g-1 \in Z_i$. We wish to show $g-1 \in Z_{i+1}$, i.e. that if $m \in A[G_{\mathbf{Q}}]$, then the coefficient $c_{i+1}((g-1)m) = 0$. To do this, we will show that $c_{i+1}((g-1)m) \in A$ occurs as a coefficient in an A-linear dependence between a set of A-linearly independent elements in A_ϕ. Consider the element:

$$x_i := (\phi - r_{n-i})(\phi - r_{n-i-1}) \dots (\phi - r_2)(g-1) = \sum_{I \subseteq \{2,\dots,n-i\}} \left(\phi^{(n-1)-|I|}(g-1) \prod_{i \in I} r_i \right)$$

in $A_\phi[G_{\mathbf{Q}}]$. By assumption $A_\phi = A_\phi^{\text{ord},\mathbf{0}}$, and hence $c_{i+1}(x_i m) = 0$ by relation 2.6. On the other hand, one may expand $c_{i+1}(x_i m)$ using Amistur's formula (formula 2.1) in terms of the alphabet

$$X := \left(\phi^{(n-1)-|I|}(g-1)m \prod_{i \in I} r_i : I \subseteq \{2, \dots, n-i\} \right).$$

Because $X \subset Z_i$, all terms in Amistur's formula vanish, except possibly those associated to the words x^{i+1} for $x \in X$. Consequently, one observes:

$$c_{i+1}(x_i m) = \sum_{I \subseteq \{2,\dots,n-i\}} \left(c_{i+1}(\phi^{(n-1)-|I|}(g-1)m) \prod_{i \in I} r_i^{i+1} \right). \tag{2.7}$$

Each of the coefficients $c_{i+1}(\phi^{(n-1)-|I|}(g-1)m)$ are elements of A, and since A_ϕ is a free A-algebra on the basis $\mathcal{B}' := (r_2^{e_2} \cdots r_n^{e_n} : 0 \leq e_j < n-j+1)$, we conclude that all such coefficients must vanish. In particular, the coefficient of 1, which is $c_{i+1}((g-1)m)$, equals 0. □

Our numerical criterion for unramifiedness follows immediately.

Proof of Theorem 2.3.1. By previous theorem, to check if a determinant $D : A[G_{\mathbf{Q}}] \to A$ is unramified at p, it suffices check that the quotient map $A_\phi \twoheadrightarrow A_\phi^{\text{ord},\mathbf{0}}$ is an isomorphism. The former algebra is free over A of rank $n!$, and the latter, by assumption, admits an A-module quotient which is free of the same rank. Hence it suffices to note that a surjection $M \to N$ of free finitely generated A-modules of the same rank is an isomorphism, which follows from [Mat86], Thm 2.4. □

In the next section, we will apply this criterion to study the determinants attached to weight one Hecke algebras.

3. Application: weight-one modular determinants are unramifed at p

In this section, we apply our criterion for unramifiedness to the study of determinants attached to modular Hecke algebras. Let $N \geq 5$ be a positive integer prime to p. Let $X_1(N)$ be the modular curve considered as a smooth proper curve over $\mathrm{Spec}(\mathbf{Z}_p)$, and let ω be the pushforward of the relative dualizing sheaf of the universal elliptic curve $\mathcal{E}/X_1(N)$. Let $\mathbf{T}_1$ be the $\mathbf{Z}_p$-subalgebra of $\mathrm{End}_{\mathbf{Z}_p}(H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p))$ generated by Hecke operators T_n and $\langle n \rangle$ for all n prime to N. Our goal in this section will be to attach a determinant $D : \mathbf{T}_1[G_{\mathbf{Q}}] \to \mathbf{T}_1$, which is unramified outside the primes dividing N, including p, such that for all primes l prime to N, the characteristic polynomial of a Frobenius element at l is:

$$P_{D,\mathrm{Frob}_l}(X) = 1 - T_l X + \langle l \rangle X^2.$$

To construct D, we will produce a determinant

$$D_1 : \mathbf{T}_1^{p\text{-adic}}[G_{\mathbf{Q}}] \to \mathbf{T}_1^{p\text{-adic}}$$

valued in a weight one p-adic Hecke algebra, which is unramified at all primes l prime to Np, and has the characterisitc polynomial

$$P_{D_1,\mathrm{Frob}_l}(X) = 1 - T_l X + \langle l \rangle X^2$$

for any Frobenius element of a prime $l \nmid Np$. The determinant D_1 will be a p-adic limit of determinants attached to classical Hecke algebras in cohomological weight and the determinant D will be the base change of D_1 to $\mathbf{T}_1$. We will then show that our criterion for unramifiedness applies to D, and the characteristic polynomial of Frob_p is as stated. As our constuctions require maps between quotients of Hecke algebras of various weights, it will be useful to regard all Hecke algebras as quotients of an abstract Hecke algebra; we denote by $\mathbf{T}$ the abstract Hecke algebra over $\mathbf{Z}_p$ generated by Hecke operators T_n and $\langle n \rangle$ for all n prime to pN. The Hecke algebra $\mathbf{T}_1$ is not *a priori* a quotient of $\mathbf{T}$. This will be a consequence of the existence of the determinant D.

We begin by recalling the definition of the p-adic Hecke algebra of weight 1. Let $X_1(N)^{ss}/\mathbf{F}_p$ denote the subscheme of $X_1(N)_{\mathbf{F}_p}$, which parameterizes supersingular elliptic curves over $\mathbf{F}_p$ with level N-structure. Associated to $X_1(N)$ is a formal scheme $X_1(N)^{fs}$ defined over $\mathbf{Z}_p$. Let $X_1(N)^{\mathrm{ord}}$ be the formal scheme theoretic complement of $X_1(N)^{ss}$ inside $X_1(N)^{fs}$. The global sections of $\omega \otimes \mathbf{Q}_p/\mathbf{Z}_p$ over $X_1(N)^{\mathrm{ord}}$ are called ($\mathbf{Q}_p/\mathbf{Z}_p$-valued) p-adic modular forms of weight 1. On $H^0(X_1(N)^{\mathrm{ord}}, \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p))$, there is an action of the Hecke algebra $\mathbf{T}$. We denote the closure of the image of $\mathbf{T}$ inside $\mathrm{End}_{\mathbf{Z}_p}(H^0(X_1(N)^{\mathrm{ord}}, \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p))$ by $\mathbf{T}_1^{p\text{-adic}}$.

Theorem 3.0.1. *There is a determinant* $D_1 : \mathbf{T}_1^{p\text{-}adic}[G_{\mathbf{Q}}] \to \mathbf{T}_1^{p\text{-}adic}$ *of degree* 2, *which is unramified at all primes* l *prime to* Np, *such that the characteristic polynomial of* Frob_l *is*

$$P_{D_1,\mathrm{Frob}_l}(X) = 1 - T_l X + \langle l \rangle X^2.$$

Proof. For each positive integer k, let $\mathbf{T}_k$ be the image of $\mathbf{T}$ inside the ring of endomorphisms $\mathrm{End}_{\mathbf{Z}_p}(H^0(X_1(N), \omega^k \otimes \mathbf{Q}_p/\mathbf{Z}_p))$. Multiplication by lifts of appropriate powers of the Hasse invariant gives the usual identification

$$\mathbf{T}_1^{p\text{-adic}} = \varprojlim \mathbf{T}_{1+(p-1)p^{k-1}}/p^k. \tag{3.1}$$

If $n > 1$, then $\mathbf{T}_n$ is a finite, flat $\mathbf{Z}_p$-algebra and the base change $\mathbf{T}_n \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is a finite étale $\mathbf{Q}_p$-algebra. The $\overline{\mathbf{Q}}_p$-points of $\mathbf{T}_n$ are in bijection with the set of classical newforms f over $\overline{\mathbf{Q}}_p$ (not necessarily cuspidal), which arise at levels dividing N. To each such point $\pi : \mathbf{T}_n \to \overline{\mathbf{Q}}_p$, there is an associated 2-dimensional $\overline{\mathbf{Q}}_p$-valued $G_{\mathbf{Q}}$-representation, which is unramified outside the primes dividing Np, such that the characteristic polynomial of Frob_l under the associated determinant is:

$$P_{D_\pi, \mathrm{Frob}_l}(X) = 1 - \pi(T_l)X + \pi(\langle l \rangle) l^{n-1} X^2.$$

The ring $\mathbf{T}_n \otimes \overline{\mathbf{Q}}_p$ is isomorphic to the sum of its $\overline{\mathbf{Q}}_p$-points, and the determinants D_π glue to yeild a determinant $D_n : \mathbf{T}_n \otimes \overline{\mathbf{Q}}_p[G_{\mathbf{Q}}] \to \mathbf{T}_n \otimes \overline{\mathbf{Q}}_p$. Since the characteristic polynomials of the elements of $G_{\mathbf{Q}}$ have coefficients valued in $\mathbf{T}_n \subseteq \mathbf{T}_n \otimes \overline{\mathbf{Q}}_p$, Amistur's formula implies that the determinant D_n decends to a $\mathbf{T}_n$-valued determinant $D_n : \mathbf{T}_n[G_{\mathbf{Q}}] \to \mathbf{T}_n$, which is unramified outside primes dividing Np, such that the characteristic polynomial of Frob_l under the associated determinant is:

$$P_{D_n, \mathrm{Frob}_l}(X) = 1 - T_l X + \langle l \rangle l^{n-1} X^2.$$

If $n = (p-1)p^{k-1} + 1$, the characteristic polynomial of Frob_l under the base change of D_n to $\mathbf{T}_n/p^k\mathbf{T}_n$ is:

$$P_{D_n, \mathbf{T}_n/p^k, \mathrm{Frob}_l}(X) = 1 - T_l X + \langle l \rangle X^2.$$

It follows that this sequence of base changes is a compatible sequence of $\mathbf{T}_{1+(p-1)p^{k-1}}/p^k$-valued determinants. Taking the inverse limit, one obtains the desired $\mathbf{T}_1^{p\text{-adic}}$-valued determinant. □

Restriction yields a $\mathbf{T}$-equivariant injection:

$$H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p)) \hookrightarrow H^0(X_1(N)^{\mathrm{ord}}, \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p)).$$

Define D to be the base change of D_1 to $\mathbf{T}_1$ along the resulting map $\mathbf{T}_1^{p\text{-adic}} \to \mathbf{T}_1$.

We claim D is unramified at p. The proof will use our criterion for unramifiedness. To apply this criterion, we must first show that D ordinary weight $\mathbf{0} := (0, 0)$, i.e. that the map $\mathbf{T}_1^{p\text{-adic}} \to \mathbf{T}_1$ factors through the quotient $(\mathbf{T}_1^{p\text{-adic}})^{\mathrm{ord}, \mathbf{0}}$. Our next theorem will show that the algebras $(\mathbf{T}_1^{p\text{-adic}})^{\mathrm{ord}, \mathbf{0}}$ and $(\mathbf{T}_1^{p\text{-adic}})_\phi^{\mathrm{ord}, \mathbf{0}}$ act on the space of *ordinary p-adic modular forms* of weight 1 in a way which is compatible with the $\mathbf{T}_1^{p\text{-adic}}$-action. Using this modular interpretation of these quotients, we will show that D_1 is split of weight $\mathbf{0}$ over $\mathrm{Spec}(\mathbf{T}_1)$.

We begin by recalling the definition of the spaces of *ordinary modular forms* of weight k and the corresponding quotients of $\mathbf{T}$. The theory of the canonical subgroup gives a well defined operator U_p on the space of ordinary modular forms. On the other hand, for weights $k > n$, the Hecke operator T_p modulo p^n coincides with U_p, and hence there is a U_p-equivariant map

$$H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p) \to \lim H^0(X_1(N), \omega^{1+(p-1)p^{k-1}} \otimes p^{-k}\mathbf{Z}_p/\mathbf{Z}_p).$$

Let $e = \lim U_p^{n!}$ be Hida's idempotent. Let $\mathbf{T}_k$ denotes the action of Hecke algebra (away from Np) on $H^0(X_1(N), \omega^k \otimes \mathbf{Q}_p/\mathbf{Z}_p)$, and let $\mathbf{T}_k^{p\text{-ord}}$ and $\widetilde{\mathbf{T}}_k^{p\text{-ord}}$ denote the images of $\mathbf{T}_k$

and $\mathbf{T}_k[T_p]$ respectively on $eH^0(X_1(N),\omega^k\otimes\mathbf{Q}_p/\mathbf{Z}_p)$. We denote the quotients of $\mathbf{T}_1^{p\text{-adic}}$ and $\mathbf{T}_1^{p\text{-adic}}[U_p]$ which act faithfully on $e(H^0(X_1(N)^{\mathrm{ord}},\omega\otimes\mathbf{Q}_p/\mathbf{Z}_p))$ by $\mathbf{T}_1^{\mathrm{Hida}}$ and $\widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$, respectively. As previously, we have isomorphisms:

$$\mathbf{T}_1^{\mathrm{Hida}}\cong\varprojlim\mathbf{T}_{1+k(p-1)p^{k-1}}^{p\text{-ord}}/p^k,\qquad \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}\cong\varprojlim\widetilde{\mathbf{T}}_{1+(p-1)p^{k-1}}^{p\text{-ord}}/p^k. \tag{3.2}$$

Theorem 3.0.2. *The base change of D_1 to $\mathbf{T}_1^{\mathrm{Hida}}$ is ordinary of weight $\mathbf{0}:=(0,0)$. An ordinary filtration for $(D_1)_{\mathbf{T}_1^{\mathrm{Hida}}}$ can be realized over $\widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$, and chosen so that the induced map*

$$\left(\mathbf{T}_1^{\mathrm{Hida}}\right)_{\phi}^{\mathrm{ord},\mathbf{0}}\to\widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$$

maps r_1 to U_p.

Proof. Like the construction of D_1, this fact will be deduced from the structure of the Galois representations attached to modular eigenforms in high weight. Let $n>1$. In the proof of Theorem 3.0.1, we constructed a determinant $D_n:\mathbf{T}_k[G_{\mathbf{Q}}]\to\mathbf{T}_n$, which was unramified outside Np, such that the Frobenius element for any l prime to Np equaled:

$$P_{D_n,\mathrm{Frob}_l}(X)=1-T_lX+\langle l\rangle l^{n-1}X^2.$$

The base change of D_1 to $\mathbf{T}_1^{\mathrm{Hida}}$ is the inverse limit of the base change of the determinants $D_{1+(p-1)p^{k-1}}$ modulo p^k via the isomorphism in equation 3.2. We claim:

Claim 2. *The base change of D_n to $\mathbf{T}_n^{p\text{-}ord}$ is ordinary of weight $(0,n-1)$, and an ordinary filtration for $(D_n)_{\mathbf{T}_n^{p\text{-}ord}}$ can be realized over $\widetilde{\mathbf{T}}_n^{p\text{-}ord}$, so that the induced map*

$$\left(\mathbf{T}_n^{p\text{-}ord}\right)_{\phi}^{\mathrm{ord},\mathbf{w}}\to\widetilde{\mathbf{T}}_n^{p\text{-}ord}$$

maps r_1 to U_p, where $\mathbf{w}=(0,n-1)$. If $n-1\equiv 0\mod p^{k-1}(p-1)$, then the base change of D_n to $\mathbf{T}_n^{p\text{-}ord}/p^n$ is ordinary of weight $\mathbf{0}$, and the corresponding ordinary filtration as above gives rise to a map

$$\left(\mathbf{T}_n^{p\text{-}ord}/p^k\right)_{\phi}^{\mathrm{ord},\mathbf{0}}\to\widetilde{\mathbf{T}}_n^{p\text{-}ord}/p^k$$

sending r_1 to U_p.

Assuming this claim, one sees from equation 3.2 that the ordinary quotient $(\mathbf{T}_1^{\mathrm{Hida}})^{\mathrm{ord},\mathbf{0}}=\mathbf{T}_1^{\mathrm{Hida}}$, and hence D_1 is ordinary of weight $\mathbf{0}$ over $\mathbf{T}_1^{\mathrm{Hida}}$. Furthermore, it follows that an ordinary filtration for D_1 is realized over $\widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$, and can be chosen so that the induced map

$$\left(\mathbf{T}_1^{\mathrm{Hida}}\right)_{\phi}^{\mathrm{ord},\mathbf{0}}\to\widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$$

maps r_1 to U_p.

We prove claim 2. Consider the base change of D_k to $\widetilde{\mathbf{T}}_k^{p\text{-ord}}\otimes\overline{\mathbf{Q}}_p$. We claim that this base change is ordinary of weight $(0,k-1)$. Since, the $\overline{\mathbf{Q}}_p$-algebra $\widetilde{\mathbf{T}}_k^{p\text{-ord}}\otimes\overline{\mathbf{Q}}_p$ is étale, it is enough to show this after base changing D_1 to any $\overline{\mathbf{Q}}_p$-point of $\widetilde{\mathbf{T}}_k^{p\text{-ord}}\otimes\overline{\mathbf{Q}}_p$. The set of such points is in bijection with ordinary Hecke eigenforms of level dividing N. By a theorem of Deligne, the Galois representations attached to such forms are ordinary in the classical sense of weight $(0,k-1)$, and ϕ acts on the unramified quotient of an ordinary flag by the image of U_p in $\overline{\mathbf{Q}}_p$. It follows that the associated determinants are ordinary of weight $(0,k-1)$, and the induced map $(\widetilde{\mathbf{T}}_k^{p\text{-ord}})_\phi^{\mathrm{ord},(0,k-1)} \to \overline{\mathbf{Q}}_p$ maps r_1 to the image of U_p. Gluing these determinants together, we obtain that D_k is ordinary of weight $(0, k-1)$ over $\widetilde{\mathbf{T}}_k^{p\text{-ord}} \otimes \overline{\mathbf{Q}}_p$, and an ordinary filtration is realized so that the induced map

$$(\widetilde{\mathbf{T}}_k^{p\text{-ord}})_\phi^{\mathrm{ord},(0,k-1)} \to \widetilde{\mathbf{T}}_k^{p\text{-ord}} \otimes \overline{\mathbf{Q}}_p \tag{3.3}$$

carries r_1 to U_p. Since the $\mathbf{T}_k^{p\text{-ord}}$-algebra $\widetilde{\mathbf{T}}_k^{p\text{-ord}} \otimes \overline{\mathbf{Q}}_p$ is faithful, we conclude D_k is ordinary over $\mathbf{T}_k^{p\text{-ord}}$. Finally, as $(\mathbf{T}_k)_\phi$ is generated over $\mathbf{T}_k$ by r_1, the map in equation (3.3) has image contained in (in fact equal to) the Hecke algebra $\widetilde{\mathbf{T}}_k^{p\text{-ord}}$. These constructions are compatible with reduction modulo p^k, establishing the second claim. ☐

Theorem 3.0.3. *The map $\mathbf{T}_1^{p\text{-}adic} \to \mathbf{T}_1$ factors through $\mathbf{T}_1^{\mathrm{Hida}}$. The $\mathbf{T}_1$-module $\mathbf{T}_1 \otimes_{\mathbf{T}_1^{\mathrm{Hida}}} \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$ is free of rank 2.*

Proof. The argument here is essentially the same as the doubling lemma (Lemma 3.15) of [CG], so we just sketch the details. The space of weight one modular forms embeds $\mathbf{T}$-equivariantly into the space weight one p-adic modular forms via the restriction map:

$$\mathrm{res} : H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p) \to H^0(X_1(N)^{\mathrm{ord}}, \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p).$$

The map res preserves q-expansions. To show that the map $\mathbf{T}_1^{p\text{-adic}} \to \mathbf{T}_1$ factors through $\mathbf{T}_1^{\mathrm{Hida}}$, we must show that U_p acts invertibly on the smallest U_p-stable subspace constaining the image of res. Consider the map

$$V := \langle p \rangle^{-1} \circ (U_p \circ \mathrm{res} - \mathrm{res} \circ T_p).$$

The map V is $\mathbf{T}$-equivariant and maps a form with q-expansion $f(q) \in \mathbf{Q}_p/\mathbf{Z}_p[[q]]$ to $f(q^p)$. By a theorem of Katz, there are no forms $f \in H^0(X, \omega \otimes \mathbf{F}_p)$ with q-expanion $f(q) \in \mathbf{F}_p[[q^p]]$, and hence there are no pair of forms $f, g \in H^0(X, \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p)$ such that $\mathrm{res} f = Vg$. It follows the sum

$$\mathrm{res} \oplus V : H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p)^2 \to H^0(X_1(N)^{\mathrm{ord}}, \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p).$$

is injective.

By a computation on q-expansions (for example), one observes that $U_p V = \mathrm{res}$. By definition,

$$U_p \circ \mathrm{res} = \mathrm{res} \circ T_p - V \langle p \rangle.$$

It follows the image of $\mathrm{res} \oplus V$ is $\mathbf{T}[U_p]$-stable, and upon identifying $H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p)^2$ with its image under $\mathrm{res} \oplus V$, the Hecke operator U_p acts on the image of $\mathrm{res} \oplus V$ by:

$$U_p = \begin{pmatrix} T_p & 1 \\ \langle p \rangle & 0 \end{pmatrix} \in GL_2(\mathbf{T}_1). \tag{3.4}$$

We conclude U_p acts invertibly on the image of $\mathrm{res} \circ V$ and commutes with the diagonal $\mathbf{T}_1$-action. Hence the map $\mathbf{T}_1^{p\text{-adic}} \to \mathbf{T}_1$ factors through $\mathbf{T}_1^{\mathrm{Hida}}$.

Given the matrix expression for U_p, the action of the polynomial ring $\mathbf{T}_1[U_p]$ on the image of $\mathrm{res} \oplus V$ factors through $\mathbf{T}_1[U_p]/U_p^2 - T_p U_p + \langle p \rangle$. We claim this quotient acts faithfully. Any further relation, would imply that there exists operators $A, B \in \mathbf{T}_1$ such that

$$A + BU_p = \begin{pmatrix} A + BT_p & B \\ \langle p \rangle B & A \end{pmatrix}$$

acts as 0 on $H^0(X_1(N), \omega \otimes \mathbf{Q}_p/\mathbf{Z}_p)^2$. An examination of the second column implies that $A = 0$ and $B = 0$, and hence no further relations exist.

Because the degree two determinant D_1 is ordinary of weight $\mathbf{0}$ over $\mathbf{T}_1^{\mathrm{Hida}}$, and there is a surjective map $\left(\mathbf{T}_1^{\mathrm{Hida}}\right)_\phi^{\mathrm{ord},\mathbf{0}} \to \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$, the $\mathbf{T}_1$-module $\mathbf{T}_1 \otimes_{\mathbf{T}_1^{\mathrm{Hida}}} \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$ is a quotient of the free rank two $\mathbf{T}_1$-module $\mathbf{T}_1 \otimes_{\mathbf{T}_1^{\mathrm{Hida}}} (\mathbf{T}_1^{\mathrm{Hida}})_\phi$. On the other hand, $\mathbf{T}_1 \otimes_{\mathbf{T}_1^{\mathrm{Hida}}} \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$ acts on the image of $\mathrm{res} \oplus V$ through the faithful action of the free $\mathbf{T}_1$-module $\mathbf{T}_1[U_p]/U_p^2 - T_pU_p + \langle p \rangle$ of rank two. We conclude $\mathbf{T}_1 \otimes_{\mathbf{T}_1^{\mathrm{Hida}}} \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$ is free $\mathbf{T}_1$-module of rank 2. $\square$

We now have:

Proof of Theorem 1.1. We first show that base change of D_1 to $\mathbf{T}_1$ is unramifed at p. By Theorems 3.0.2 and 3.0.3, the base change of D_1 to $\mathbf{T}_1^{\mathrm{Hida}}$ and $\mathbf{T}_1$ are both ordinary of weight zero, and hence there are identifications

$$\mathbf{T}_1^{\mathrm{Hida}} = \left(\mathbf{T}_1^{\mathrm{Hida}}\right)^{\mathrm{ord},\mathbf{0}}, \qquad \mathbf{T}_1 = (\mathbf{T}_1)^{\mathrm{ord},\mathbf{0}},$$

where the latter algebra is a quotient of the former. On the other hand, Theorem 3.0.2 also realizes $\widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$ as a quotient of $\left(\mathbf{T}_1^{\mathrm{Hida}}\right)_\phi^{\mathrm{ord},\mathbf{0}}$, and hence

$$\mathbf{T}_1 \otimes_{\mathbf{T}_1^{\mathrm{Hida}}} \widetilde{\mathbf{T}}_1^{\mathrm{Hida}}$$

as a quotient of $(\mathbf{T}_1)_\phi^{\mathrm{ord},\mathbf{0}}$. Since the module above is free of rank two by Theorem 3.0.3, it follows that $(\mathbf{T}_1)_\phi^{\mathrm{ord},\mathbf{0}}$ admits a quotient which is free of rank 2. Hence, from Theorem 2.3.1, we deduce that D_1 is unramified at p. Let us now compute the characteristic polynomial of Frob_p. By the compatibility of Theorem 3.0.2, the characteristic polynomial of Frob_p coincides with the characteristic polynomial of U_p. By equation 3.4, this is $X^2 - T_pX + \langle p \rangle$. $\square$

4. Upper Triangular Determinants

If l is a prime exactly dividing N, local-global compatibility dictates that the p-adic Galois representation attached to a classical Hecke eigenform f of weight $k \geq 2$ will be *upper triangular at* l i.e. the p-adic Galois representation $V_{p,f}/\overline{\mathbf{Q}}_p$ associated f will will admit[5] a complete $G_{\mathbf{Q}_l}$-stable flag $0 \subset F_1V_{p,f} \subset V_{p,f}$. Furthermore, such a flag can be chosen so that the $G_{\mathbf{Q}_l}$ action on the quotient $V_{p,f}/F_1V_{p,f}$ is unramified and Frob_l acts by multiplication by the eigenvalue of U_l on f. For the determinant attached to a weight one Hecke algebra, an interpolation of this property holds.

Let l prime, $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbf{Q}_p$, and $\tau_l = (\chi_1, \ldots, \chi_n)$ be an ordered collection of continuous inertial characters $\chi_i^{(0)} : I_l \to \mathcal{O}^\times$. We say a n-dimensional $G_{\mathbf{Q}}$-representation $V/\overline{\mathbf{Q}}_p$ is upper triangular at l if there exists a complete $G_{\mathbf{Q}_l}$-stable flag

$$0 = F_0V \subseteq F_1V \subseteq \ldots \subseteq F_nV = V$$

such that I_l acts on F_iV/F_{i-1} through $\chi_i^{(0)}$. Ordinary representations are a special case of upper triangular determinants. The interpolation of ordinariness given in section 2.2 generalizes straight forwardly to an interpolation of upper triangularity.

[5]For many such modular forms f, this upper–triangular representation will be split. By local–global compatibility, the representation will be non-split if and only if the local component π_l associated to f is an unramified twist of the Steinberg representation, or, equivalently, if f is new at l and the Nebentypus character of f is unramified at l.

Definition 4.0.1. *Let A be a $\mathcal{O}$-algebra. Let $D : A[G_{\mathbf{Q}}] \to A$ be a determinant of degree n, we say D is upper triangular at l with inertial graded representation τ_l, if there exists a faithful A-algebra B and an ordered collection characters $\chi_1, \ldots, \chi_n : W_l \to B^\times$ such that:*

(1) $\chi_i|_{I_l} = \chi_i^{(0)}$,
(2) *for all $g \in W_l$ the characteristic polynomial of g is:*

$$P_{D,g}(X) = \prod_{i=1}^{n}(1 - \chi_i(g)X),$$

(3) *for each positive integer $i \leq n$, sequence of elements $g_1 \ldots g_i \in W_l$, and $x \in B[G_K]$:*

$$c_j((g_i - \chi_i(g_i))(g_{i-1} - \chi_{i-1}(g_{i-1})) \cdots (g_1 - \chi_1(g_1))x) = 0,$$

if $j > n - i$.

A choice of ordered collection of characters $(\chi_1, \ldots, \chi_n)$ satisfying the above conditions is called a local graded determinant at l of type $(\chi_1^{(0)}, \ldots, \chi_n^{(0)})$.

Furthermore, each of the propositions of section 2 generalize to upper triangular determinants. In particular, there is a bijection between upper triangular determinants over $\overline{\mathbf{Q}}_p$, and upper triangular representations over $\overline{\mathbf{Q}}_p$ which preserves inertial semi-simplifications.

An argument analogous to that used to prove Theorem 3.0.2 shows that if l exactly divides N, then D_1 is upper triangular at l. Let $T \subset S$ be the finite set of primes which exactly divide N. Define $\mathbf{T}_1^{\mathrm{ss}}$ to be the $\mathbf{Z}_p$-subalgebra of $\mathrm{End}_{\mathbf{Z}_p}(H^0(X_1(N), \omega^k \otimes \mathbf{Q}_p/\mathbf{Z}_p))$ generated by the operators U_l for $l \in T$ and the Hecke operators T_n and $\langle n \rangle$ for all n prime to N. We regard the characteristic polynomial coefficient $c_2 : G_{\mathbf{Q}} \to A^\times$ of D_1 (the classical determinant) as an A-valued character. We denote the unramified, $\mathbf{T}_1^{\mathrm{ss}}$-valued character on $G_{\mathbf{Q}_\ell}$ that maps Frob_l to U_l by $\chi_l(U_l)$. One may show:

Theorem 4.0.2. *If l is a prime divides N, then the determinant $D_1 : \mathbf{T}_1[G_{\mathbf{Q}}] \to \mathbf{T}_1$ is upper triangular at l with inertial graded representation $(1, c_2)$. A $G_{\mathbf{Q}_l}$-stable filtration with local type $(\chi_l(U_l), \chi_l(U_l)^{-1}c_2)$ can be realized over $\mathbf{T}_1^{\mathrm{ss}}$.*

Suppose that N is square-free, and $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is an odd, semi-simple, 2-dimensional Galois representation of Serre conductor dividing N which is unramified at p. If $\overline{\rho}$ is irreducible, then $\overline{\rho}$ is modular [KW09a, KW09b]. Moreover, by [Gro90, CV92] the corresponding determinant associated to $\overline{\rho}$ gives rise to a finite collection of $\overline{\mathbf{F}}_p$-points of $\mathbf{T}_1^{\mathrm{ss}}$, the exact number depending on a choice of ordering of Frobenius eigenvalues for each prime dividing Np such that $\overline{\rho}$ is unramified. Alternatively, if $\overline{\rho}$ is reducible, then $\overline{\rho}$ gives rise to quotients of $\mathbf{T}_1^{\mathrm{ss}}$ via weight one Eisenstein series and their associated oldforms of level N. We are naturally led to conjecture that $\mathbf{T}_1^{\mathrm{ss}}$ is the universal $\mathbf{Z}_p$-algebra of determinants interpolating these properties.

Theorem 4.0.3. *Let N be a square-free positive integer. Let $\mathcal{F}$ be the set valued functor on Artinian $\mathbf{Z}_p$-algebras which assigns to an algebra A the set of all pairs $(D, (\tau_l : l \in T))$ consisting of:*

(1) *a continous determinant $D : A[G_{\mathbf{Q}}] \to A$ which is:*
  (a) *unramified outside the primes dividing N,*
  (b) *upper triangular at l with inertial graded representation $(1, c_{2,D})$ for all $l \mid N$,*
  (c) *odd, i.e. the characteristic polynomial $P_{D,c}(X) = 1 + X^2$ for any complex conjugation $c \in G_{\mathbf{Q}}$, and*

(2) *a tuple of A-valued local graded determinants* τ_l *at* l *of type* $(1, c_{2,D})$ *which are compatible with* D.

Then $\mathcal{F}$ *is pro-representable by a semi-local noetherian* $\mathbf{Z}_p$*-algebra* R_N^{ss}.

Proof. For each fixed residual determinant $\overline{D}$ valued in the finite field k there is, by Proposition 3.3 of [Che14], a corresponding local $W(k)$-algebra $R_{2,N,\overline{D}}$ representing continuous determinants of degree 2 deforming $\overline{D}$ which are unramified outside N and valued in p-adically complete $W(k)$-algebras. If k was minimal with respect to $\overline{D}$, then $R_{2,N,\overline{D}}$ as a $\mathbf{Z}_p$-algebra represents the same functor for $\overline{D}$ and all its conjugates, except now with respect to complete $\mathbf{Z}_p$-algebras. By Serre's conjecture, we know that $\mathcal{F}(\overline{\mathbf{F}}_p)$ is a finite set (this accounts for the irreducible determinants, the reducible determinants can be accounted for by class field theory). Let $R_{2,N}$ be the direct sum of $R_{2,N,\overline{D}}$ over each $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ conjugacy class of $\overline{D}$ in $\mathcal{F}(\overline{\mathbf{F}}_p)$.

The proof of the existence of R_N^{ss} is constructive and is analogous to that of $A_\phi^{\mathrm{ord},\mathbf{w}}$ (see section 2.2, specifically Theorem 2.2.4). Specifically, R_N^{ss} is defined by imposing relations on the $R_{2,N}$-algebra that (simulateously) parameterizes complete ordered sets of roots for the characteristic polynomials of some choices of Frobenii elements for the primes $l \in T$. This later ring is finite over $R_{2,N}$, and hence R_N^{ss} is semi-local and Noetherian. □

From Theorem 4.0.2, there is a map $\varphi_1 : R_N^{\mathrm{ss}} \to \mathbf{T}_1^{\mathrm{ss}}$ which corresponds to the pair consisting of the determinant D_1 and local types $(\chi_l(U_l), \chi_l(U_l)^{-1}c_2)$ for $l \| N$. This map is surjective. We make the following natural conjecture:

Conjecture 4.0.4. *Assume* N *is a squarefree integer. The map* $\varphi_1 : R_N^{\mathrm{ss}} \to \mathbf{T}_1^{\mathrm{ss}}$ *is an isomorphism.*

By [Cal15], one knows that if $p > 2$, then the map φ_1 is an isomorphism away from the Eisenstein locus.

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