THERE EXIST NON-CM HILBERT MODULAR FORMS OF PARTIAL WEIGHT ONE

RICHARD MOY AND JOEL SPECTER

Abstract. In this note we prove that there exists a conjugate pair of classical Hilbert modular cuspidal eigenforms over $\mathbf{Q}(\sqrt{5})$ of partial weight one which do *not* arise from the induction of a Grössencharacter from a CM extension of $\mathbf{Q}(\sqrt{5})$.

1. Introduction

A Hilbert modular form is said to be of *partial weight one* if some, but not all, of its weights are one. It is a well-established "folklore question"[1] whether there exists a totally real field F and a classical Hilbert modular form f of partial weight one which does *not* arise from the induction of a Grössencharacter from some CM extension of F. This note answers the question in the affirmative (see Theorem 1.1). If, in addition, all the weights of f have the same parity (which our conjugate pair of examples do), then, assuming local-global compatibility, there exists a compatible family of representations $(L, \{\rho_\lambda\})$ with the following intriguing property:

> Let ℓ be a prime in $\mathcal{O}_F$ not dividing the level of f and totally split in F. If λ is a prime in $\mathcal{O}_L$ above ℓ, then the corresponding representation,
> $$\rho_\lambda : G_F \to \mathrm{GL}_2(\mathcal{O}_\lambda)$$
> will be geometric, have Zariski dense image, and yet be unramified for at least one $v|\ell$.

Many cases of local-global compatibility are now known. *See, e.g.,* [8, §3.2], [9, Theorem 1.4]. Although such a beast seems somewhat peculiar, there is no obvious *a priori* reason why it should not exist. On the other hand, there does not seem to be any obvious way (even conjecturally) to produce such a modular form, either by automorphic or motivic methods. To answer the question we find an explict form. Although (in principle) the method of computation used in this paper applies to general totally real fields, we shall restrict to real quadratic fields F with narrow class number one for convenience. Indeed, all of our computations took place with Hilbert modular forms for the field $F = \mathbf{Q}(\sqrt{5})$.

2010 *Mathematics Subject Classification.* 11F41, 11F80.

Corresponding Author: Richard Moy, Northwestern University, 2033 Sheridan Road, Evanston, IL, United States. Email: ramoy88@math.northwestern.edu.

[1]We originally learnt of this problem through Fred Diamond. In conversations with Kevin Buzzard, Don Blasius, and Fred Diamond, it became clear that the question of whether such forms existed was apparent to the authors of [1] in the '80s (and may well have occurred to others before then). The question gained some urgency with the advent of Fraser Jarvis' construction of Galois representations for partial weight one forms [7] in the mid-'90s, since, if the only such forms were CM, then [7] would be a trivial consequence of Class field theory. We have heard several reports of the question being raised again at this time. In light of these stories, we feel safe in calling the problem a "well-known folklore question."

1.1. **The Computation.** Our search for partial weight one Hilbert modular forms is premised on the philosophy that finite-dimensional spaces of meromorphic modular forms which are stable under the action of the Hecke algebra ought to be modular. In the case of classical modular forms, this idea has been formalized by George Schaeffer. Let V be a finite-dimensional space of meromorphic modular forms on $\Gamma_0(N)$ of weight k and nebentypus χ which are holomorphic at infinity. In his thesis [10, Theorem 6.2.1] and in [11, Theorem 1.1], Schaeffer proves that if V is stable under the action of a Hecke operator T_p for $p \nmid N$, then $V \subseteq M_k(\Gamma_0(N), \chi, \mathbf{C})$. As a corollary, one observes that for any such V containing $M_k(\Gamma_0(N), \chi, \mathbf{C})$, the chain (for $p \not| N$)

$$V \supseteq V \cap T_p V \supseteq V \cap T_p V \cap T_p^2 V \supseteq \ldots$$

stabilizes to $M_k(\Gamma_0(N), \chi, \mathbf{C})$ in less than $\dim_{\mathbf{C}} V$ steps [10, Algorithm 7.2.6].

Schaeffer's principal application of this theorem is the effective computation of the space $M_1(\Gamma_0(N), \chi, \mathbf{C})$ of classical weight one modular forms. Suppose one wishes to compute this space. To begin, simply take any Eisenstein series $E \in M_1(\Gamma_0(N), \chi^{-1}, \mathbf{C})$ and let V be the space of ratios of forms in $M_2(\Gamma_0(N), \mathbf{C})$ with E. Then $V \supseteq M_1(\Gamma_0(N), \chi, \mathbf{C})$. It then suffices to compute the intersection of V with its Hecke translates. One can reduce this computation to one in linear algebra by passing to Fourier expansions. The Fourier expansions of forms in $M_2(\Gamma_0(N), \mathbf{C})$ are easily calculated to any bound via modular symbols and the Fourier expansion of E has a simple formula. Hence, the Fourier expansion of any form in V is easily calculated to any bound. The operator T_p acts on Fourier expansions formally via a well-known formula. What makes the method effective is that it requires only an explict finite number of Fourier coefficients for a basis of the space V to calculate $M_1(\Gamma_0(N), \chi, \mathbf{C})$. The number of coefficients required is determined by the Sturm bounds.

Schaeffer's method generalizes nicely to the case of Hilbert modular forms. Let $\mathfrak{n}$ be a modulus of $\mathbf{Q}(\sqrt{5})$ and χ a totally odd ray class character of conductor $\mathfrak{n}$. Fix an odd integer $m > 1$. We are interested in calculating the space $S_{[m,1]}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ of Hilbert cusp forms of partial weight one. As in the case of classical modular forms, there exists an Eisenstein series $E_{1,\chi^{-1}} \in M_{[1,1]}(\Gamma_0(\mathfrak{n}), \chi^{-1}, \mathbf{C})$ and one can consider the space V of ratios with numerators in $S_{[m+1,2]}(\Gamma_0(\mathfrak{n}), \mathbf{C})$ and denominator $E_{1,\chi^{-1}}$. This is a finite-dimensional space of meromorphic forms which contains $S_{[m,1]}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ as its maximal holomorphic subspace. Assuming $\mathfrak{n}$ is square-free, one can use Dembélé's algorithm [4, 5] as implemented in `magma` [2], to produce the Fourier expansions of a basis for the space $S_{[m+1,2]}(\Gamma_0(\mathfrak{n}), \mathbf{Q}(\sqrt{5}))$ to any desired degree of accuracy. The Fourier expansion of $E_{1,\chi^{-1}}$ is given by an explicit formula. Hence, the Fourier expansion of the meromorphic forms in V can be calculated to any desired degree of accuracy. For a prime $\mathfrak{p}$ of $\mathcal{O}_F$, the Hecke operator $T_{\mathfrak{p}}$ acts on the Fourier expansions of the meromorphic forms in V formally via an explicit formula. So, as in the case of classical forms, one may hope to calculate the $T_{\mathfrak{p}}$-stable subspace of V via techniques in linear algebra.

Unfortunately, this direct generalization of Schaeffer's method is impractical from a computational perspective. In comparison with the case of classical modular forms, the number of Fourier coefficients needed to prove equality of two modular forms and the amount of computation needed to calculate those Fourier coefficients is much greater. For this reason, we structure our search method so that it requires as few Fourier coefficients as possible.

For the details of our search, we refer the reader to Section 2.6. But the idea is as follows; we calculate the Fourier expansions of the forms in $S_{[m+1,2]}(\Gamma_0(\mathfrak{n}), 1)/E_{1,\chi^{-1}}$ truncated to some chosen bound. We calculate the intersection of these spaces of truncated formal Fourier expansions using linear algebra. If the dimension of the intersection coincides with the number of eigenforms with complex multiplication (CM), then every eigenform in $S_{[n,1]}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{Q}(\sqrt{5}))$ has CM. (Using class field theory, we can compute the dimension of the CM subspace in advance.) In practice, using a modest bound, we were able to restrict the existence of a non-CM weight one Hilbert modular form of small level and norm to a handful of candidate spaces where the intersection is larger than expected. One can then check if a form $f \in V$ in such a candidate space is holomorphic by checking if there exists a form $g \in S_{[dn,d]}(\Gamma_0(\mathfrak{n}), \chi^d, \mathbf{Q}(\sqrt{5}))$ such that the $f^d = g$. Our search yielded the existence of a nonparallel weight one Hilbert modular form without CM.

Main Theorem. *Let $\mathfrak{n} = (14) \subset \mathcal{O}_{\mathbf{Q}(\sqrt{5})}$ and let χ be the order 6 ray class character of conductor $(7) \cdot \infty_1\infty_2$ such that $\chi(2) = \frac{-1+\sqrt{-3}}{2}$. The space of cusp forms $S_{[5,1]}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ is 2-dimensional, and has a basis with coefficients in $\mathbf{Q}(\sqrt{5}, \chi)$. This space has a basis over $\mathbf{Q}(\sqrt{5}, \chi, \sqrt{-19})$ consisting of two conjugate eigenforms, neither of which admit complex multiplication.*

Remark 1.1. Let π be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ associated to either of these newforms. Implimenting Schaeffer's algorithm for a second time, one finds that the space of forms $S_{[5,1]}(\Gamma_0(7), \chi)$ is empty. Hence the level of π at 2 is $\Gamma_0(2)$. Since the character χ has conductor prime to 2 and the level at 2 is $\Gamma_0(2)$, the local component π_2 is Steinberg (up to an unramified quadratic twist). In particular, this implies that local-global compatibility results of [8, 9] could not be proved directly using congruence methods to higher weight, which would only be sufficient for proving compatibility up to N-semi-simplification.

2. Hilbert Modular Forms

In this section, we state some basic definitions and results on classical Hilbert modular forms. Let F be a real quadratic field of narrow class number one and $\mathcal{O}_F$ its ring of integers. We fix an ordering on the two embeddings of F into $\mathbf{R}$ and denote, for $a \in F$, the image of a under the i-th embedding by a_i. We say an element $a \in \mathcal{O}_F$ (resp. $a \in F$) is totally positive if $a_i > 0$ for all i and denote the set of all such elements by $\mathcal{O}_F^+$ (resp. F^+). Similarly, we have two natural embeddings of the matrix ring $\mathrm{M}_2(F)$ into the matrix ring $\mathrm{M}_2(\mathbf{R})$. If $\gamma \in \mathrm{M}_2(F)$, let γ_1 and γ_2 denote the image of γ under the i-th embedding. Let $\mathfrak{d}_F = (\delta)$ be the different of $F/\mathbf{Q}$ where $\delta \in \mathcal{O}_F^+$. For an integral ideal $\mathfrak{n}$ of F, we define

$$\Gamma_0(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) : a, d \in \mathcal{O}_F, \quad c \in \mathfrak{n}\mathfrak{d}, \quad b \in \mathfrak{d}^{-1}, \quad ad - bc \in \mathcal{O}_F^\times \right\}$$

where $\mathrm{GL}_2^+(F)$ is the subgroup of $\mathrm{GL}_2(F)$ composed of matrices with totally positive determinant. If $\mathbf{H}$ is the complex upper half-plane, the group $\Gamma_0(\mathfrak{n})$ acts on $\mathbf{H} \times \mathbf{H}$ via fractional linear transformations by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .(z_1, z_2) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right).$$

Let $\underline{k} := [k_1, k_2]$ be an ordered pair of nonnegative integers. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F)$ and $z \in \mathbf{H} \times \mathbf{H}$ set

$$j(\gamma, z)^{\underline{k}} := \det(\gamma_1)^{\frac{-k_1}{2}} \det(\gamma_2)^{\frac{-k_2}{2}} (c_1 z_1 + d_1)^{k_1} (c_2 z_2 + d_2)^{k_2}.$$

If $f : \mathbf{H} \times \mathbf{H} \to \mathbf{C}$ and $\gamma \in \mathrm{GL}_2^+(F)$, we write $f|_\gamma$ to mean the function $f|_\gamma : \mathbf{H} \times \mathbf{H} \to \mathbf{C}$ given by

$$f|_\gamma(z) = j(\gamma, z)^{-\underline{k}} f(\gamma z). \tag{2.1}$$

Consider a character $\chi : (\mathcal{O}_F/\mathfrak{n})^\times \to \mathbf{C}^\times$ which satisfies $\chi(u) = \left(\frac{u_1}{|u_1|}\right)^{-k_1} \left(\frac{u_2}{|u_2|}\right)^{-k_2}$ for all $u \in \mathcal{O}_F^\times$. A *Hilbert modular form* of weight $\underline{k}$, level $\mathfrak{n}$, and character χ is a holomorphic function $f \colon \mathbf{H} \times \mathbf{H} \to \mathbf{C}$ such that for all $\gamma \in \Gamma_0(\mathfrak{n})$,

$$f|_\gamma(z) = \chi(d) f(z). \tag{2.2}$$

We denote the $\mathbf{C}$-vector space of all such functions by $M_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ and by $M_{\underline{k}}(\Gamma_0(\mathfrak{n}), \mathbf{C})$ when χ is the trivial character. As in the case of classical modular forms, we can compute Fourier expansions of Hilbert modular forms.

2.1. **Fourier Expansions.** If $f \in M_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$, then for all $d \in \mathfrak{d}_F^{-1}$

$$f(z) = f(z + d)$$

by the transformation rule (2.2) since $\left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$. It follows from Fourier analysis that the form f is given by the series

$$f(z) = \sum_{\alpha \in \mathcal{O}_F} c_\alpha(f) e^{2\pi i(\alpha_1 z_1 + \alpha_2 z_2)}$$

in a neighborhood of the cusp (∞, ∞). The Koecher Principle [6, p. 18] states that $c_\alpha(f) = 0$ unless α is totally positive or $\alpha = 0$. If the constant term of the Fourier expansion of $f|_\gamma$ is zero for all $\gamma \in \mathrm{GL}_2^+(F)$, then we call f a *cusp form* and denote the space of such forms $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$. We denote the space of cusp forms of level $\mathfrak{n}$, weight $\underline{k}$, and trivial character by $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \mathbf{C})$.

Besides the Koecher Principle, the Fourier expansions of Hilbert modular forms have additional structure. Let $f \in S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$. For any totally positive unit η in $\mathcal{O}_F$, one can check that the coefficient $c_\alpha(f)$ satisfies the identity:

$$c_{\eta\alpha}(f) = \eta_1^{k_1/2} \cdot \eta_2^{k_2/2} \cdot c_\alpha(f) = \eta_2^{(k_2 - k_1)/2} \cdot c_\alpha(f) \tag{2.3}$$

by using the transformation rule (2.2) with $\left(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ and equating Fourier expansions. If desired, one can create a Fourier expansion indexed over the ideals of F rather than indexed over elements of $\mathcal{O}_F$. In particular, for an ideal $\mathfrak{a} = (\alpha)$, we can set

$$c(\mathfrak{a}, f) := N(\mathfrak{a})^{(k_1 - k_2)/2} \cdot c_\alpha(f)/\alpha_1^{(k_1 - k_2)/2} = c_\alpha(f) \cdot {\alpha_2}^{(k_1 - k_2)/2}, \tag{2.4}$$

and one can easily check that this is independent of the choice of totally positive generator α of $\mathfrak{a}$ by using (2.3) above. We will call the $c_\alpha(f)$ the *unnormalized* Fourier coefficient of f and we will call the $c(\mathfrak{a}, f)$ the *normalized* Fourier coefficients of f as in [3, p. 458]. Observe that $c(\mathfrak{a}, f) = c_\alpha(f)$ if $k_1 = k_2$.

2.2. **Hecke Operators.** For an integral ideal $\mathfrak{n}$ of $\mathcal{O}_F$, let

$$\Gamma_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) : a \in \mathcal{O}_F, \quad b \in \mathfrak{d}^{-1}, \quad c \in \mathfrak{n}\mathfrak{d}, \quad d-1 \in \mathfrak{n} \right\}.$$

If $\mathfrak{q}$ is an integral ideal of $\mathcal{O}_F$, we may choose a totally positive generator π of $\mathfrak{q}$ and write the disjoint union

$$\Gamma_1(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(\mathfrak{n}) = \coprod_j \Gamma_1(\mathfrak{n})\gamma_j$$

where the γ_j are a finite set of right coset representatives. We define the $\mathfrak{q}^{th}$ *Hecke operator* by

$$T_{\mathfrak{q}} f := \sum_j f|_{\gamma_j}. \tag{2.5}$$

The action of $T_{\mathfrak{q}}$ on spaces of modular forms is independent of our choice of totally positive generator for $\mathfrak{q}$. To see this let $\eta \in (\mathcal{O}_F^+)^\times$ and observe that

$$\Gamma_1(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & \eta\pi \end{pmatrix} \Gamma_1(\mathfrak{n}) = \Gamma_1(\mathfrak{n}) \begin{pmatrix} \eta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \Gamma_1(\mathfrak{n}) = \coprod_j \Gamma_1(\mathfrak{n})\gamma_j \left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta \end{smallmatrix}\right).$$

Let $\tilde{\gamma}_j = \gamma_j \left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta \end{smallmatrix}\right)$. Using (2.1), it is easy to verify that $f|_{\gamma_j} = f|_{\tilde{\gamma}_j}$. Hence, the action of $T_{\mathfrak{q}}$ does not depend on our choice of a totally positive generator π of $\mathfrak{q}$.

If $\mathfrak{q} = (\pi)$ is a prime ideal relatively prime to $\mathfrak{n}$, then we may choose the following coset representatives for the γ_j:

$$\gamma_\beta := \begin{pmatrix} 1 & \epsilon\delta^{-1} \\ 0 & \pi \end{pmatrix} \quad \text{and} \quad \gamma_\infty := \begin{pmatrix} \alpha & \beta\delta^{-1} \\ \delta\nu & \pi \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}.$$

where ϵ runs through a complete set of representatives for $\mathcal{O}_F/\mathfrak{n}$, δ is a totally positive generator for the different $\mathfrak{d}$, ν is a totally positive generator for $\mathfrak{n}$, and $\alpha, \beta \in \mathcal{O}_F$ such that $\alpha\pi - \nu\beta = 1$. Let $f \in M_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ have Fourier expansion $f = \sum_{\alpha \in \mathcal{O}_F} c_\alpha e^{2\pi i(\alpha_1 z_1 + \alpha_2 z_2)}$, then $T_{\mathfrak{q}}$ has the following effect on the Fourier expansion of the modular form f:

$$c_\alpha(T_{\mathfrak{q}} f) = \pi_1^{\frac{-k_1}{2}+1} \pi_2^{\frac{-k_2}{2}+1} c_{\alpha\pi} + \pi_1^{\frac{k_1}{2}} \pi_2^{\frac{k_2}{2}} \chi(\mathfrak{q}) c_{\alpha/\pi} = \pi_2^{\frac{k_1}{2}-\frac{k_2}{2}} N(\mathfrak{q})^{-\frac{k_1}{2}+1} c_{\alpha\pi} + \pi_2^{\frac{k_2}{2}-\frac{k_1}{2}} N(\mathfrak{q})^{\frac{k_1}{2}} \chi(\mathfrak{q}) c_{\alpha/\pi}$$

where $N(\mathfrak{q})$ denotes the numerical norm of the ideal $\mathfrak{q}$ and $c_{\alpha/\pi} = 0$ if $\alpha/\pi \notin \mathcal{O}_F$. On the other hand, if $\mathfrak{q}$ is prime and divides $\mathfrak{n}$, then

$$c_\alpha(T_{\mathfrak{q}} f) = \pi_1^{\frac{-k_1}{2}+1} \pi_2^{\frac{-k_2}{2}+1} c_{\alpha\pi}(f).$$

2.3. **Basis For $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$.** In general, there will not be a basis of eigenforms for $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$. Rather, there will be a *newspace* $S_{\underline{k}}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ which will be generated by eigenforms which we now describe.

Let $\mathfrak{m}$ be a divisor of $\mathfrak{n}$, and let $\mathfrak{b}$ be a divisor of $\mathfrak{n}/\mathfrak{m}$. Then there is a map

$$V_{\mathfrak{m},\mathfrak{b}} : S_{\underline{k}}(\Gamma_0(\mathfrak{m}), \chi, \mathbf{C}) \to S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$$

given by

$$\sum_{\alpha \in \mathcal{O}_F} c_\alpha q^\alpha \mapsto \sum_{\alpha \in \mathcal{O}_F} c_\alpha q^{b\alpha}$$

where $\mathfrak{b} = (b)$ and $b \in \mathcal{O}_F^+$. This map only depends on b up to a scalar which one can easily verify from (2.3). Let $S_{\underline{k}}^{\text{old}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ be the subspace of $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ spanned by $V_{\mathfrak{m},\mathfrak{b}}(f)$ for all $f \in S_{\underline{k}}(\Gamma_0(\mathfrak{m}), \chi, \mathbf{C})$ and all $(\mathfrak{m}, \mathfrak{b})$ with $\mathfrak{m}|\mathfrak{n}$ where $\mathfrak{m} \neq \mathfrak{n}$ and $\mathfrak{b}|(\mathfrak{n}/\mathfrak{m})$. The orthogonal complement of $S_{\underline{k}}^{\text{old}}(\Gamma_0(\mathfrak{n}), \mathbf{C})$, under the Petersson inner product, is the space $S_{\underline{k}}^{\text{new}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$; it has a basis of eigenforms which we will refer to as *newforms.*

For weights $\underline{k}$ such that all $k_i \in \underline{k}$ are positive even integers, Dembélé's algorithm computes the space of newforms $S_{\underline{k}}^{\text{new}}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ by using the fact that they are in bijection, via the Jacquet-Langlands correspondence, with a certain space of automorphic forms on a quaternion algebra. We then exploit the fact, special to GL(2), that the Fourier expansion of a newform can be recovered from its Hecke eigenvalues.

2.4. **Eisenstein Series of Weight One.** In [12, Proposition 3.4], Shimura gives a prescription which attaches to certains pairs of narrow ideal class characters of a number field F, with $[F : \mathbf{Q}] > 1$, an Eisenstein series of parallel weight k. The Fourier expansions of these Eisenstein series are calculated in [3, Proposition 2.1], and we recall this result here. As we only make use of Eisenstein series of parallel weight $\underline{k} = [1, 1]$ associated to pairs consisting of a trivial and nontrivial character, we include only the details which are relevant to this case.

In the classical setting, Eisenstein series of weight one are constructed via analytically continuating a two variable Eisenstein series defined via Hecke summation and then specializing the continued variable at the origin. In general, this procedure produces a function on the upper half plane with the correct monodromy under the apppropriate modular group but which might fail to be holomorphic. In the setting of Hilbert modular varieties, the analogous construction always defines a holomorphic Hilbert modular form. Explicitly, let ψ be a totally odd character of the narrow ray class group modulo $\mathfrak{n}$ and let

$$U = \{u \in \mathcal{O}_F^\times : \text{Nm}(u) = 1, \ u \equiv 1 \mod \mathfrak{n}\}.$$

For $z \in \mathbf{H}^2$, $s \in \mathbf{C}$ with $\text{Re}(2s+1) > 2$, and $e_F(x) = \exp(2\pi i \cdot \text{Tr}_{F/\mathbf{Q}}(x))$, define

$$f(z,s) := C \cdot \frac{1}{\text{Nm}(\mathfrak{n})} \sum_{\substack{a \in \mathcal{O}_F, \ b \in \mathfrak{d}^{-1} \\ (a,b) \mod U, \ (a,b) \neq (0,0)}} \left(\frac{1}{(az+b)|az+b|^{2s}} \times \sum_{c \in \mathcal{O}_F/\mathfrak{n}} \text{sgn}(c)^{[1,1]} \psi(c) e_F(-bc) \right)$$

where

$$C := \frac{\sqrt{d_F}}{[\mathcal{O}_F^\times : U]\text{Nm}(\mathfrak{d})(-2\pi i)^2}$$

and $\text{sgn}(c)^r := \text{sgn}(c_1)^{r_1}\text{sgn}(c_2)^{r_2}$ and $r = [r_1, r_2] \in (\mathbf{Z}/2\mathbf{Z})^2$.

Observe that the above sum for $f(z,s)$ is over pairs (a,b) of nonzero elements of the product $\mathcal{O}_f \times \mathfrak{d}^{-1}$ modulo the action of U (which is diagonal multiplication) as well as over the representatives c for $\mathcal{O}_F/\mathfrak{n}$.

For fixed z, $f(z,s)$ has meromorphic continuation in s to the entire complex plane. Set

$$E_{1,\psi}(z) := f(z,0).$$

In [3, Proposition 2.1], the authors compute the normalized Fourier coefficients of the above Eisenstein series, $E_{1,\psi}$. However, the normalized and unnormalized coefficients are equal here since the Eisenstein series has parallel weight. Their result is summarized in the following proposition.

Proposition 2.1. *Let $\mathfrak{n} \neq \mathcal{O}_F$ be an integral ideal of F and let ψ be a totally odd character of the narrow ray class group modulo $\mathfrak{n}$. Then there exists an element $E_{1,\psi} \in M_{[1,1]}(\Gamma_0(\mathfrak{n}), \psi, \mathbf{C})$ such that $c_\alpha(E_{1,\psi}) = \sum_{\mathfrak{m}|(\alpha)} \psi(\mathfrak{m})$ for all totally positive integers α of $\mathcal{O}_F^+$ and $c_0(E_{1,\psi}) = \frac{L(\psi,0)}{4}$. Explicitly,*

$$E_{1,\psi} = \frac{L(\psi,0)}{4} + \sum_{b \in \mathcal{O}_F^+} \left(\sum_{\mathfrak{m}|(b)} \psi(\mathfrak{m}) \right) \cdot e_F(bz)$$

2.5. **CM Forms.** While in general spaces of Hilbert modular forms of partial weight one are mysterious, we do have one source to reliably produce such forms; we can obtain them via automorphic induction from certain Grössencharacters. Specifically, let K be a totally imaginary quadratic extension of F and $\mathbb{A}_K$ be the adeles of K. Let $\nu_1 : K \to \mathbf{C}$ be an embedding extending the place ∞_1. Consider a Grössencharacter

$$\psi : \mathrm{GL}_1(K)\backslash\mathrm{GL}_1(\mathbb{A}_K) \to \mathbf{C}^\times$$

such that the local components of ψ at the infinite places are

$$\psi_{\infty_1}(z) = z_{\nu_1}^{k-1} \qquad \text{and} \qquad \psi_{\infty_2}(z) = |z|_{\infty_2}^{k-1}.$$

Then, by a theorem of Yoshida [13], there exists a unique Hilbert modular eigenform f_ψ of weight $[k,1]$ such that the L-function of f_ψ is equal to the L-function of ψ. The form f_ψ is independent of the choice of extension of ∞_1.

A Hilbert modular eigenform f is said to have CM if its primitive form is equal to f_ψ for some character ψ. From the equality of L-functions, one observes that if $\mathfrak{p}$ is a prime of F which is inert in K, then the normalized Hecke eigenvalue $c(\mathfrak{p}, f_\psi) = 0$. Conversely, this property classifies CM Hilbert modular forms. That is, if f is a Hilbert modular form of level $\mathfrak{c}$ and K is a totally imaginary extension of F such that $c(\mathfrak{p}, f) = 0$ for all primes $\mathfrak{p} \nmid \mathfrak{c}$ which are inert in K, the primitive form of f is f_ψ for some Grössencharacter ψ of K. By class field theory, one can restate this fact as follows.

Theorem 2.2. *Let f be a Hilbert modular eigenform of level $\mathfrak{c}$. Then f has CM if and only if there exists a totally odd quadratic Hecke character ϵ of F of conductor $\mathfrak{f}$ such that $c(\mathfrak{p}, f)\epsilon(\mathfrak{p}) = c(\mathfrak{p}, f)$ for all $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{f}$. In this case, we say f has CM by ϵ.*

If f_ψ is a newform arising from the character ψ, then the level of f is equal to $\Delta_{K/F} N_{K/F}(\mathfrak{f}(\psi))$ where $f(\psi)$ is the conductor of ψ. It follows that if f is a CM form of level $\Gamma_1(\mathfrak{c})$, then f has CM by some Hecke character of conductor dividing $\mathfrak{c}$. There are only finitely many such Hecke characters, and so one can verify by calculating finitely many Hecke eigenvalues of f that f does not have CM.

2.6. **The Algorithm.** In this section, we outline the algorithm used to search for non-CM modular forms of weight $[k,1]$.

Recall from Section 2.1, that the nonzero coefficients appearing in the Fourier expansion of a Hilbert modular form are indexed by the totally nonnegative elements of $\mathcal{O}_F$. Fix a field L and consider the ring of formal Fourier expansions over L (coefficients indexed by the totally nonnegative elements of $\mathcal{O}_F$). For any pair of integers $B := (b_1, b_2)$ there is an ideal of this ring consisting of all formal Fourier series whose Fourier coefficient $c_\alpha = 0$ if $|\alpha|_{\infty_1} < b_1$ and $|\alpha|_{\infty_2} < b_2$. The ring of formal Fourier expansions (over L) truncated to the bound B is defined to be the quotient of the ring of formal Fourier expansions by this ideal.

Algorithm 1. *The following is a procedure to search for weight $[k,1]$ modular forms. Which on input $(\underline{k}, \mathfrak{n}, \chi, B, \mathfrak{q})$ consisting of*

(1) $\underline{k} = [k,1]$ *a pair of odd integers,*
(2) $\mathfrak{n}$ *a square-free integral ideal of* F,
(3) χ *a totally odd ray class character of* F *of conductor dividing* $\mathfrak{n} \cdot \infty_1\infty_2$,
(4) $B = (b_1, b_2)$ *a pair of positive integers,*
(5) $\mathfrak{q}$ *a prime ideal of* $\mathcal{O}_F$

outputs a pair, $(\delta, V^{\mathfrak{q}}(B))$, *consisting of a natural number* δ *which bounds the number of non-CM eigenforms from above and a finite dimensional space,* $V^{\mathfrak{q}}(B)$, *of formal Fourier expansions truncated to the bound* B *which contains the image of* $S_{[k,1]}(\Gamma_0(\mathfrak{n}), \chi, F(\chi))$.

(1) Using Dembélé's algorithm [4, 5] (see Section 2.3), compute, for each $\mathfrak{m}|\mathfrak{n}$, a basis for the image of $S^{\text{new}}_{[k+1,2]}(\Gamma_0(\mathfrak{m}), F)$ in the ring of formal Fourier expansions over $F(\chi)$ truncated to the bound $N_{F/\mathbf{Q}}(\mathfrak{q}) \cdot B$.
(2) Using the bases calculated in step 1 and following the procedure described in Section 2.3, compute a basis $S_{[k+1,2]}(N_{F/\mathbf{Q}}(\mathfrak{q}){\cdot}B, F)$ (resp. $S_{[k+1,2]}(B)$) for the image of $S_{[k+1,2]}(\Gamma_0(\mathfrak{n}), F)$ in the ring of formal Fourier expansions over F truncated to the bound $N_{F/\mathbf{Q}}(\mathfrak{q}) \cdot B$ (resp. B). If the cardinality of $|S_{[k+1,2]}(B)| < \dim(S_{[k+1,2]}(\Gamma_0(\mathfrak{n}), F))$, then terminate the algorithm and display the error message "Bound B is insufficient."
(3) Divide the truncated Fourier expansions in $S_{[k+1,2]}(N_{F/\mathbf{Q}}(\mathfrak{q}) \cdot B)$ (resp.$S_{[k+1,2]}(B)$) by the Fourier expansion for $E_{1,\chi^{-1}}$ described in Section 2.4. Call the resulting set of truncated formal Fourier expansions $\tilde{S}_{[k,1]}(N_{F/\mathbf{Q}}(\mathfrak{q}) \cdot B)$ (resp. $\tilde{S}_{[k,1]}(B)$). Let $V^{\emptyset}(B)$ be the space spanned by $\tilde{S}_{[k,1]}(B)$.
(4) Compute $T_{\mathfrak{q}}f$ for each element f of $\tilde{S}_{[k,1]}(N_{F/\mathbf{Q}}(\mathfrak{q}) \cdot B)$. The result is a set of formal Fourier expansions truncated to the bound B. Let $T_{\mathfrak{q}}V^{\emptyset}(B)$ be the space spanned by this set.
(5) Compute bases for $V^{\emptyset}(B) \cap T_{\mathfrak{q}}V^{\emptyset}(B)$ and $\ker\left((T_{\mathfrak{q}} \colon V^{\emptyset}(N_{F/\mathbf{Q}}(\mathfrak{q}) \cdot B) \to T_{\mathfrak{q}}V^{\emptyset}(B)\right)$. Let $V^{\mathfrak{q}}(B)$ be the space spanned by the union of these bases.
(6) Compute the dimension of the subspace in $S_{[k,1]}(\Gamma_0(\mathfrak{n}), \chi, \mathbf{C})$ spanned by CM forms using class field theory. Denote this dimension by h.
(7) Let $\delta = \dim(V^{\mathfrak{q}}(B)) - h$. Return $(\delta, V^{\mathfrak{q}}(B))$. □

If the algorithm returns an output with $\delta = 0$, then all the eigenforms in the space $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi)$ are CM. If the algorithm returns an output with $\delta > 0$, one increases the bound B and reruns the algorithm. If δ stabilizes at some value greater 0 over several increases in precision, this indicates there might exist a non-CM eigenform of weight $\underline{k}$ in the space $S_{\underline{k}}(\Gamma_0(\mathfrak{n}), \chi)$.

All of our calculations were made for $F = \mathbf{Q}(\sqrt{5})$. We first used the algorithm to calculate the dimensions of the spaces $M_{[3,1]}(\Gamma_0(\mathfrak{n}), \chi)$ where $\mathfrak{n}$ is a square-free ideal of $\mathcal{O}_F$ and χ is a totally odd character modulo $\mathfrak{n}$. We restricted ourselves to the case where $\mathfrak{n}$ is square-free, because the `magma` package used only worked in this case. Our program searched through all square-free $\mathfrak{n}$ of norm less than 500 and quadratic χ, but we did not find any non-CM Hilbert modular forms. (In fact, our calculations show that none exist in the spaces we computed).

We next used our algorithm to calculate dimensions of $M_{[5,1]}(\Gamma_0(\mathfrak{n}), \chi)$ for all square-free ideals $\mathfrak{n}$ of norm less than 300. The only candidate space our algorithm found is described below in Section 3. In all other spaces of modular forms, our algorithm found that all eigenforms were CM.

3. A non CM form

Let $F = \mathbf{Q}(\sqrt{5})$. We order the infinite places of F such that $|\sqrt{5}|_{\infty_1} > 0$. The ray class group of conductor $(7)\cdot\infty_1\infty_2$ is isomorphic to $\mathbf{Z}/6\mathbf{Z}$. Let χ be the order 6 character such that $\chi(2) = \frac{-1+\sqrt{-3}}{2}$. The character χ is totally odd.

Theorem 3.1. *The space of cusp forms $S_{[5,1]}(\Gamma_0(14), \chi, \mathbf{C})$ is 2-dimensional and has a basis with coefficients in $F(\chi)$. This space has a basis over $F(\chi, \sqrt{-19})$ consisting of two conjugate eigenforms, neither of which admit complex multiplication.*

Proof. For n a positive integer, we define

$$b(n) := \left(\left\lfloor \frac{5n - \sqrt{5}n}{2} \right\rfloor, \left\lfloor \frac{5n + \sqrt{5}n}{2} \right\rfloor \right)$$

where $\lfloor\cdot\rfloor$ denotes the floor function. Applying Algorithm 1 with input $(\underline{k}, \mathfrak{n}, \chi, B, (2)) = ([5,1], 14\mathcal{O}, \chi, B, (2))$ with $B = b(24), b(26)$ and $b(28)$, respectively, one finds that for each value $V^{(2)}(B)$ is two-dimensional. Table 1 lists the initial normalized Fourier coefficients of one of the truncated formal Fourier expansions in $V^{(2)}(B)$. Let $f(q)$ be this formal Fourier expansion in $V^{(2)}(28)$. Let $f \in S_{[6,2]}(\Gamma_0(14), 1, F)/E_{1,\chi^{-1}}$ be a meromorphic modular form whose Fourier expansion truncated to the bound $b(28)$ is $f(q)$. We show $f \in S_{[5,1]}(\Gamma_0(14), \chi, F(\chi))$, by showing $f^3 \in S_{[15,3]}(\Gamma_0(14), \chi^3, F)$. This is done in two steps.

(1) First we show the truncation map taking a form in $S_{[18,6]}(\Gamma_0(14), F)$ to its Fourier expansion truncated to the bound $b(28)$ is an injection.
(2) Next we find a form $g \in S_{[15,3]}(\Gamma_0(14), \chi^3, F)$ such that the Fourier expansions of g and f are equivalent when truncated to the bound $b(28)$.

Noting that $(f^3 - g)E^3_{1,\chi} \in S_{[18,6]}(\Gamma_0(14), F)$, it follows from (1) and (2) that f^3 and g are equal.

The proofs of facts (1) and (2) are both computational. Using the `magma` package, one computes that the space of cusp forms $S_{[18,6]}(\Gamma_0(14), F)$ has dimension 356. Then one computes explicitly the Fourier expansions for a basis of $S_{[18,6]}(\Gamma_0(14), F)$ truncated to the bound $b(28)$ and shows that the resulting set of truncated formal Fourier series span a space of the same dimension. This proves (1).

To prove (2), one must construct an element $S_{[15,3]}(\Gamma_0(14), \chi^3, F)$ with a desired property. Unfortunately, the creation of spaces of Hilbert modular forms with nontrivial nebentypus and the computation of their Fourier expansions has not yet been implemented in the `magma` package. To skirt this issue, we instead use the `magma` package to compute the Fourier expansions of the 56-dimensional space $S_{[14,2]}(\Gamma_0(14), F)$ truncated to the bound $b(56)$. One then obtains the Fourier expansions for the forms in the subspace

$$E_{1,\chi^3}.S_{[14,2]}(\Gamma_0(14), F) + T_2(E_{1,\chi^3}.S_{[14,2]}(\Gamma_0(14), F)) \subseteq S_{[15,3]}(\Gamma_0(14), \chi^3, F)$$

truncated to the bound $b(28)$, in which, following a calculation in linear algebra, one finds a form g as desired in (2). It follows $S_{[5,1]}(\Gamma_0(14), \chi, \mathbf{C})$ is 2-dimensional and has a basis with elements in $F(\chi)$.

We now demonstrate the second claim of the proposition: that $S_{[5,1]}(\Gamma_0(14), \chi, \mathbf{C})$ has a basis over $F(\chi, \sqrt{-19})$ consisting of two conjugate eigenforms, neither of which admit complex multiplication. Utilizing Algorithm 1, one computes that $V^{(2)}([5,1], 7\mathcal{O}, \chi, b(28), (2)) = 0$

and hence $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C}) = S^{\mathrm{new}}_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$. It follows $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$ has a basis over $\mathbf{C}$ of simultaneous eigenforms for the Hecke algebra. As $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$ has a basis defined over $F(\chi)$ and is two-dimensional, these eigenforms have as a field of definition either $F(\chi)$ or a quadratic extension of $F(\chi)$. Calculating the characteristic polynomial of T_5 on $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$, we obtain that the field of definition is $F(\chi,\sqrt{-19})$.

Finally, we see that neither of the forms in $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$ are CM. If this were not the case, both forms of $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$ would have CM by a quadratic character of conductor 14. The unique such character is χ^3. However, one observes that $\chi^3(\frac{7+\sqrt{5}}{2}) = -1$ and the $\frac{7+\sqrt{5}}{2}$ normalized Hecke eigenvalue does not vanish for either eigenform in $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$. □

Remark 3.2. *The Galois group* $\mathrm{Gal}(F(\chi,\sqrt{-19})/\mathbf{Q}) = (\mathbf{Z}/2\mathbf{Z})^3$ *acts on the Fourier expansion as follows. The element with fixed field* $F(\chi)$ *permutes the two eigenforms. The element with fixed field* $\mathbf{Q}(\sqrt{5},\sqrt{-19})$ *sends the eigenform to an eigenform in* $S_{[5,1]}(\Gamma_0(14),\chi^{-1},\mathbf{C})$*, where* χ^{-1} *is the conjugate of* χ*. The element with fixed field* $\mathbf{Q}(\sqrt{-3},\sqrt{-19})$ *sends the eigenform to a form in* $S_{[1,5]}(\Gamma_0(14),\chi,\mathbf{C})$*.*

See Table 1 for the normalized coefficients $c(\mathfrak{p})$ for various prime ideals $\mathfrak{p} = (\pi)$ of small norm for one of the two normalized eigenforms in $S_{[5,1]}(\Gamma_0(14),\chi,\mathbf{C})$. If $c(\pi)$ is a coefficient in the Fourier expansion of our eigenform for a prime π, then the normalized coefficient is $c(\mathfrak{p}) = c(\pi)\overline{\pi}^2$ as seen in (2.4). The normalized coefficient does not depend on the choice of totally positive generator π for the ideal $\mathfrak{p} = (\pi)$.

Table 1. Table of Normalized Coefficients of Eigenform in $S_{[5,1]}(\Gamma_0(14),\chi)$

π	$N(\pi)$	$c(\mathfrak{p}),\ \mathfrak{p} = (\pi)$
2	4	$-4+4\sqrt{-3}$
$\frac{5+\sqrt{5}}{2}$	5	$\dfrac{-45+15\sqrt{-3}+15\sqrt{-19}-15\sqrt{57}}{4}$
3	9	$-18-18\sqrt{-3}-9\sqrt{-19}\left(\dfrac{3-\sqrt{-3}}{2}\right)$
$\frac{7+\sqrt{5}}{2}$	11	$\dfrac{-87+87\sqrt{-3}+36\sqrt{5}-36\sqrt{-15}+63\sqrt{-19}-21\sqrt{57}+24\sqrt{-95}-8\sqrt{285}}{4}$
$\frac{9+\sqrt{5}}{2}$	19	$\dfrac{-456+152\sqrt{-3}+171\sqrt{5}-57\sqrt{-15}+66\sqrt{-19}-66\sqrt{57}-39\sqrt{-95}+39\sqrt{285}}{4}$
$\frac{11+\sqrt{5}}{2}$	29	$-162+\dfrac{417}{2}\sqrt{5}+66\sqrt{57}+\dfrac{17}{2}\sqrt{285}$
$\frac{13+\sqrt{5}}{2}$	41	$\left(49+12\sqrt{5}\right)\cdot\left(\dfrac{9\sqrt{-3}+15\sqrt{-19}}{2}\right)$
7	49	$\dfrac{-1715+1715\sqrt{-3}+1029\sqrt{-19}+1029\sqrt{57}}{4}$

Remark 3.3. We checked that for $N(\mathfrak{p}) < 1000$ and $\gcd(N(\mathfrak{p}),14) = 1$ the Satake parameters of π satisfy the Ramanujan Conjecture. Equivalently, the Hecke eigenvalues satisfy the bounds $|c(\mathfrak{p})|_{\infty_1} \le 2p^2$ and $|c(\mathfrak{p})|_{\infty_2} \le 2p^2$. The Ramanujan conjecture would follow from Deligne's proof of the Riemann hypothesis if one knew that π was *motivic*, however, the construction of the associated Galois representations proceeds via congruences.

4. Acknowledgements

The authors would like to extend special thanks to Frank Calegari for introducing this problem, his excellent advising, and his extensive comments on preliminary drafts of this paper. The authors would also like to thank Kevin Buzzard, Lassina Dembélé, and James Newton for their comments on a preliminary version of this paper, and Don Blasius, Kevin Buzzard, and Fred Diamond for their historical remarks.

5. Funding

The first author was supported in part by National Science Foundation Grant DMS-1404620. The second author was supported in part by National Science Foundation Grant DMS-1404620 and by a National Science Foundation Graduate Research Fellowship under Grant No. DGE-1324585.

References

[1] Blasius, Don, and Jon Rogawski. "Galois representations for Hilbert modular forms." *Bulletin of the American Mathematical Society* 21, no. 1 (1989): 65–69.

[2] Bosma, Wieb, John Cannon, and Catherine Playoust, "The Magma algebra system I. The user language." *Journal of Symbolic Computation* 24 (1997): 235–65.

[3] Dasgupta, Samit, Henri Darmon, and Robert Pollack, "Hilbert modular forms and the Gross-Stark conjecture." *Annals of Mathematics* 174 (2011): 439–84.

[4] Dembélé, Lassina. "Explicit computations of Hilbert modular forms on $\mathbb{Q}\sqrt{5}$." *Experimental Mathematics* 14, no. 4 (2005): 457–66.

[5] Dembélé, Lassina. "Quaternionic Manin symbols, Brandt matrices and Hilbert modular forms." *Mathematics of Computation* 76, no. 258 (2007): 1039–57.

[6] Van der Geer, Gerard. *Hilbert Modular Surfaces*. Berlin: Springer-Verlag, 1988.

[7] Jarvis, Frazer. "On Galois representations associated to Hilbert modular forms." *Journal für die reine und angewandte Mathematik* 491 (1997): 199-216.

[8] Luu, Martin. "Deformation Theory and Local-Global Compatibility of Langlands Correspondences." *Memoirs of the American Mathematical Society* (forthcoming).

[9] Newton, James. "Local-global compatibility for Galois representations associated to Hilbert modular forms of low weight." Preprint.

[10] Schaeffer, George. "The Hecke stability method and ethereal forms." PhD diss., University of California, Berkeley, 2012.

[11] Schaeffer, George. "Hecke stability and weight 1 modular forms." Preprint. http://arxiv.org/abs/1406.0408.

[12] Shimura, Goro. "The special values of the zeta functions associated with Hilbert modular forms." *Duke Mathematical Journal* 45, no. 3 (1978): 637–79.

[13] Yoshida, Hiroyuki. "On the representations of the galois groups obtained from Hilbert modular forms." PhD diss., Princeton University, 1973.

Richard Moy, Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, United States

E-mail address: `ramoy88@math.northwestern.edu`

Joel Specter, Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, United States

E-mail address: `jspecter@math.northwestern.edu`