Minimal Polynomial of a diagonal matrix

Then: let
\[
D = \begin{pmatrix}
    \lambda_1 & & 0 \\
    & \ddots & \\
    0 & & \lambda_n
\end{pmatrix}
\]
be a diagonal matrix,

\[\text{and } \lambda_1, \ldots, \lambda_n \text{ be the eigenvalues of } D \text{ (the diagonal entries of } D \text{ included without multiplicity — i.e. each}
\]
\[\lambda_i \text{ is different from } \lambda_j \text{ for all } i \neq j), \text{ then the minimal polynomial of } D \text{ is:}
\]
\[
m_D(x) = (x - \lambda_1) \ldots (x - \lambda_n).
\]

In particular, \( m_D(x) \) has no repeated roots.

Example:
\[
D = \begin{pmatrix}
    2 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 2
\end{pmatrix}, \quad \{\lambda_1, \lambda_2, \lambda_3\} = \{1, 2\}
\]
\[
m_D(x) = (x - 1)(x - 2).
\]
pf: Let \( f(x) = (x-A_1) \cdots (x-A_k) \) and \( e_i \) be a standard basis vector. We will show \( f(D)e_i = 0 \).

Observe:

\[
f(D)e_i = (D-A_1I) \cdots (D-A_kI)e_i \\
= \left[ (D-A_1I) \cdots (D-A_{k-1}I) \right] (D-A_kI)e_i \\
= \left[ (D-A_1I) \cdots (D-A_{k-1}I) \right] (v_i - A_ke_i) \\
= \left[ (D-A_1I) \cdots (D-A_{k-1}I) \right] (v_i - A_k)e_i
\]

(2)

Since \( (v_i - A_k) \) is a scalar, it commutes with matrix multiplication, so:

\[
(2) = (v_i - A_k) \left[ (D-A_1I) \cdots (D-A_{k-1}I) \right] e_i
\]

Repeating steps (1) \( \cdots \) (2) with the next \( k-1 \) factors gives:

\[
(v_i - A_k)[(D-A_1I) \cdots (D-A_{k-1}I)] e_i \\
= (v_i - A_k)(v_i - A_{k-1}) \cdots (v_i - A_1)e_i \\
= f(v_i)e_i
\]

(On eigenvectors acting by polynomials in \( D \) is the same as acting by polynomials in \( v_i \))

Since, \( v_i = A_j \) (for some \( j \))

\( v_i \) is a root of \( f(x) \); hence \( f(D)e_i = 0 \). So \( \ker \left( f(D) \right) \) contains a basis, and hence \( f(D) \) is the zero matrix. Since each \( A_i \) is an eigenvalue, and hence a root of the minimal polynomial, we conclude \( f(x) = p(x) \)