Practice Final II

Problem 1.  (1) Find equations for three planes \( P_1, P_2, P_3 \) (not necessarily through the origin) with the properties that:
   (a) none of the planes are parallel to the coordinate planes,
   (b) the intersection of the planes \( P_1 \cap P_2 \cap P_3 \) is empty,
   (c) for any pair of planes the intersection \( P_i \cap P_j \) is nonempty.
(2) Find a linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) such that the image of \( P_1 \) and \( P_2 \) under \( T \) are parallel to coordinate planes. Is it possible to find a linear transformation so that all three planes are parallel to coordinate planes?
(3) Define \textbf{rank}. If you were to encode your equations from part 1 as a matrix equation \( Ax = b \) what would be the rank of \( A \)? Would any answer yield a matrix of the same rank?

Problem 2. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation such that

\[
T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \quad T \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

(1) What is the determinant of \( T \)?
(2) Find a vector \( w \) such that \( T^9 w = e_3 \). Is \( w \) unique?
(3) Determine the characteristic polynomial of \( T \).

Problem 3 (120 points; suggested time: 14-25 minutes). Let

\[
A := \begin{bmatrix} 15 & -12 & 0 \\ 12 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}
\]

Circle the matrices below which are similar to \( A \). If a matrix below is not similar to \( A \) give a reason why not.
Problem 4. Given a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, one way to measure its size is to identify it with a vector in $\mathbb{R}^4$. Define

$$||A||_{\text{coeff}} := \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Find the best approximation to the left and right inverses to the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, i.e. find a pair of matrices $L_A$ and $R_A$ which minimize $||L_A A - I||_{\text{coeff}}$ and $||A R_A - I||_{\text{coeff}}$, respectively. Are $R_A$ and $L_A$ unique?

Problem 5. (1) Find a non-scalar $3 \times 3$ matrix $A$ which is both orthogonal and symmetric with the property that $\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$ is an eigenvector for $A$. Up to similarity how many such matrices are there?

(2) Describe the transformation given by multiplication by $A$ geometrically.
(3) Is it possible to find a matrix $A$ satisfying the conditions in part (1) with the additional property that
\[
\begin{bmatrix}
22 \\
41 \\
101
\end{bmatrix}
\] is an eigenvector? Why or why not?