Shear Problem Solution

1. Definition: An \( n \times n \) matrix \( E \) is elementary if it can be obtained from \( I_n \) by performing one of the three elementary row operations on \( I_n \).

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

are elementary matrices.

Elementary row operations can be realized by multiplying the corresponding elementary matrices on the left.

E.g. Multiplying a \( 2 \times m \) matrix by \( \begin{bmatrix} 1 & k & 0 \end{bmatrix} \) on the left amounts to adding \( k \) times the second row of that matrix to its first row.

Multiplying a \( 2 \times m \) matrix by \( \begin{bmatrix} 1 & 0 & k \end{bmatrix} \) on the left amounts to adding \( k \) times the first row of that matrix to its second row.

2. (a) \[
\begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
2
\end{bmatrix} =
\begin{bmatrix}
1 + 2k \\
2
\end{bmatrix}
\]

Letting \( t \) vary, we see that the set is the line \( x_2 = 2 \).

Similarly:

(b) \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} \quad \text{Set } x_1 = 1
\]

(c) \[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
x_3
\end{bmatrix} \quad \text{Set } x_3 = 1
(f) Note that if the vector is on a coordinate axis, i.e. one of its components is 0, the situation is a bit different:

(g) We see from the above calculation that a point with no zero component (not on an axis) can move horizontally or vertically when multiplied by an appropriate upper or lower triangular shear. A point on the $x_1$-axis can only move vertically, and a point on the $x_2$-axis can only move horizontally, when multiplied by an appropriate shear.

Therefore, if a point (= vector) is not on the $x_i$-axis, we can first move it horizontally to $[1]$ by multiplying it by an appropriate upper triangular shear, and then move $[1]$ to $[0]$ by multiplying by an appropriate lower triangular shear. $[1]$ vertically.
The process is illustrated by the picture below.

If however, the point is on the $x$-axis, then recall that it can only move vertically. So we first move it vertically away from the $x$-axis, then horizontally to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and finally vertically to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Each step amounts to multiplying by a shear. The process is illustrated in the picture below.

By the above analysis, at most 3 shears are needed to convert a nonzero vector to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(4) Observe that the determinant of any (upper or lower) shear is 1.

Then $S_1 S_2 \ldots S_n A_1 = A_2$

$\Rightarrow \det(S_1 S_2 \ldots S_n A_1) = \det(A_2)$

$\Rightarrow \det(S_1) \det(S_2) \ldots \det(S_n) \det(A_1) = \det(A_2)$

$\Rightarrow \det(A_1) = \det(A_2)$.

Hence, two shear equivalent matrices have the same determinant.

(5) Suppose $A = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$. Then for any 2x2 matrix $S$, $SA = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$.

Since $A$ is invertible, the first column of $A$ is a nonzero vector. We may first multiply $A$ on the left by a sequence of shears so that the first column of $A$ becomes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (see (3) above).

Thus, $A$ is transformed to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since shear equivalent
matrices have the same determinant, it follows that $b_2 = \det(A)$. Then, we may multiply $\begin{bmatrix} 1 & b_1 \\ 0 & \det(A) \end{bmatrix}$ on the left by an appropriate upper triangular shear (this amounts to adding a multiple of the second row to the first row) to cancel the $b_1$. In this way, we bring $A$ to $\begin{bmatrix} 1 & 0 \\ 0 & \det(A) \end{bmatrix}$.

(6) No, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are two matrices with determinant 0, but they are not shear equivalent. In fact, no matter what shear you multiply $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with, the second column is always $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so you can never bring $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ into $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Any two matrices $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & l \\ 0 & 0 \end{bmatrix}$, $k \neq l$, are not shear equivalent. No matter how we multiply by shears, in the first matrix, the second column is always $k$ times the first column, while in the second matrix, the second column is always $l$ times the first column. Hence, there are infinitely many matrices of determinant 0 up to shear equivalence.