

THE CRYSTALLINE PERIOD OF A HEIGHT ONE p -ADIC DYNAMICAL SYSTEM

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ABSTRACT. Let f be a continuous ring endomorphism of $\mathbf{Z}_p[[x]]/\mathbf{Z}_p$ of degree p . We prove that if f acts on the tangent space at 0 by a uniformizer and commutes with an automorphism of infinite order, then it is necessarily an endomorphism of a formal group over \mathbf{Z}_p . The proof relies on finding a stable embedding of $\mathbf{Z}_p[[x]]$ in Fontaine's crystalline period ring with the property that f appears in the monoid of endomorphisms generated by the Galois group of \mathbf{Q}_p and crystalline Frobenius. Our result verifies, over \mathbf{Z}_p , the height one case of a conjecture by Lubin.

1. INTRODUCTION

Given a one-dimensional formal group law F/\mathbf{Z}_p , the endomorphism ring of F provides an example of a nontrivial family of power series over \mathbf{Z}_p which commute under composition. This paper investigates the question of the converse: under what conditions do families (in our case pairs) of commuting power series arise as endomorphisms of an integral formal group? Let p be a prime number, \mathbf{C}_p be the completion of an algebraic closure of \mathbf{Q}_p , and $\mathfrak{m}_{\mathbf{C}_p} \subset \mathbf{C}_p$ be the open unit disk. Define $S_0(\mathbf{Z}_p)$ to be the set of formal power series over \mathbf{Z}_p without constant term.¹ Substitution defines a composition law on $S_0(\mathbf{Z}_p)$ under which it is isomorphic to the monoid of endomorphisms of the formal p -adic unit disk which fix 0.

Main Theorem. *Let f and u be a commuting pair of elements in $S_0(\mathbf{Z}_p)$. If*

- $f'(0)$ is prime in \mathbf{Z}_p and f has exactly p roots in $\mathfrak{m}_{\mathbf{C}_p}$,
- u is invertible and has infinite order,

then there exists a unique formal group law F/\mathbf{Z}_p such that $f, u \in \text{End}_{\mathbf{Z}_p}(F)$. The formal group law F is isomorphic to $\widehat{\mathbb{G}}_m$ over the ring of integers of the maximal unramified extension of \mathbf{Q}_p .

The study of analytic endomorphisms of the p -adic disk was initiated by Lubin in [Lub94]. There he showed that if $f \in S_0(\mathbf{Z}_p)$ is a non-invertible transformation which is nonzero modulo p and $u \in S_0(\mathbf{Z}_p)$ is an invertible, non-torsion transformation, then assuming f and u commute:

- (1) The number of roots of f in $\mathfrak{m}_{\mathbf{C}_p}$ is a power of p (cf. [Lub94, Main Theorem 6.3, p. 343]).

2010 *Mathematics Subject Classification.* 11S20, 11S31, 11S82 (Primary), 14L05, 13F25, 14F30 (Secondary).

Key words and phrases. p -adic dynamical system, p -adic Hodge Theory, Lubin-Tate Group.

The author was supported in part by National Science Foundation Grant DMS-1404620 and by a National Science Foundation Graduate Research Fellowship under Grant No. DGE-1324585.

¹In general, for any ring R , we define $S_0(R)$ to be the set formal of power series over R without constant term.

- (2) The reduction of f modulo p is of the form $a(x^{p^h})$ where $a \in S_0(\mathbf{F}_p)$ is invertible (cf. [Lub94, Corollary 6.2.1, p. 343]).²

Both (1) and (2) are well known properties of a non-invertible endomorphism of a formal group law over \mathbf{Z}_p . In light of this, and knowing:

- (3) If a power series $u \in S_0(\mathbf{Q}_p)$ acts on the tangent space at 0 as an automorphism with infinite order, then it is an automorphism of a unique one dimensional formal group law F_u/\mathbf{Q}_p . If $f \in S_0(\mathbf{Q}_p)$ commutes with u then $f \in \text{End}_{\mathbf{Q}_p}(F_u)$ (cf. [Lub94, Theorem 1.3, p. 327]),

one might preliminarily conjecture that the formal group F_u associated to u is defined over \mathbf{Z}_p . This is false and counterexamples have been constructed by Lubin [Lub94, p. 344] and Li [Li02, p. 86-87]. Instead, Lubin hypothesizes that “for an invertible series to commute with a non-invertible series, there must be a formal group somehow in the background” [Lub94, p. 341]. Interestingly, all known counterexamples are constructed from the initial data of a formal group defined over a finite extension of \mathbf{Z}_p and therefore conform to Lubin’s philosophy. Despite this, the “somehow” in Lubin’s statement has yet to be made precise.

Result (2) is reminiscent of a hypothesis in the following integrality criterion of Lubin and Tate:

- (4) Let $f \in S_0(\mathbf{Z}_p)$. Assume $f'(0)$ is prime in \mathbf{Z}_p and $f(x) \equiv x^{p^h} \pmod{p}$, then there is a unique formal group law F_f/\mathbf{Z}_p such that $f \in \text{End}_{\mathbf{Z}_p}(F)$ (cf. [LT65, Lemma 1, p. 381-2]).

Perhaps based on this and the counterexamples of [Li02] and [Lub94], Lubin has offered the following conjecture as to when our preliminary intuition should hold true:

Lubin’s Conjecture. [Sar05, p. 131] *Suppose that f and u are a non-invertible and a non-torsion invertible series, respectively, defined over the ring of integers \mathcal{O} of a finite extension of \mathbf{Q}_p . Suppose further that the roots of f and all of its iterates are simple, and that $f'(0)$ is a uniformizer in \mathbf{Z}_p . If $f \circ u = u \circ f$ then $f, u \in \text{End}_{\mathcal{O}}(F)$ for some formal group law F/\mathcal{O} .*

For each integral extension of \mathbf{Z}_p , Lubin’s conjecture is naturally divided into cases: one for each possible height of the commuting pair f and u . This note serves as a complete solution to the height one case of Lubin’s conjecture over \mathbf{Z}_p . Previous results towards Lubin’s conjecture ([LMS02], [Sar10], [SS13]) have used the field of norms equivalence to prove some special classes of height one commuting pairs over \mathbf{Z}_p arise as endomorphisms of integral formal groups. All other cases of Lubin’s conjecture, i.e. those over proper integral extensions of \mathbf{Z}_p or over \mathbf{Z}_p with height greater than one, are completely open.

We call a formal group Lubin-Tate (over \mathbf{Z}_p) if it admits an integral endomorphism f satisfying the hypotheses of (4). All height one formal groups over \mathbf{Z}_p are Lubin-Tate. Therefore, given the height one commuting pair (f, u) , one strategy to prove that the formal group F_u is integral is to identify the Lubin-Tate endomorphism of F_u and appeal to result (4). This is the approach we employ in this paper.

Before we present an outline of the proof, we remind the reader of some structures which exist in the presence of a formal group. Let F be a height one formal group over \mathbf{Z}_p . Then F is Lubin-Tate, so there exists a unique endomorphism f of F such that $f(x) \equiv x^p \pmod{p}$.

²The analogous statements both hold when \mathbf{Z}_p is replaced by the ring of integers in a finite extension of \mathbf{Q}_p .

Consider the \mathfrak{f} -adic Tate module $T_{\mathfrak{f}}F$. The absolute Galois group of \mathbf{Q}_p acts on $T_{\mathfrak{f}}F$ through a character $\chi_F : G_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_p^\times$.

The character χ_F is crystalline. We sketch a proof of this fact following [KR09]. Let $\mathcal{O}_{\mathbf{C}_p}$ be the integral closure of \mathbf{Z}_p in \mathbf{C}_p and $\tilde{\mathbf{E}}^+ := \varprojlim \mathcal{O}_{\mathbf{C}_p}/p$, where the limit is taken with respect to the p -power Frobenius map. Because $\mathfrak{f}(x) \equiv x^p \pmod{p}$, elements of $T_{\mathfrak{f}}$ are canonically identified with elements of $\tilde{\mathbf{E}}^+$. Let v be a generator of $T_{\mathfrak{f}}$. There is a unique element $\hat{v} \in \tilde{\mathbf{A}}^+ := W(\tilde{\mathbf{E}}^+)$, the Witt vectors of $\tilde{\mathbf{E}}^+$, which lifts v and with the property that the action of $G_{\mathbf{Q}_p}$ and the Frobenius endomorphism φ satisfy:

- (A) $g(\hat{v}) = [\chi_F(g)]_F(\hat{v})$ for all $g \in G_{\mathbf{Q}_p}$,
- (B) $\varphi(\hat{v}) = \mathfrak{f}(\hat{v})$ [KR09, Lemma 1.2].

The logarithm of F evaluated at \hat{v} converges in Fontaine's crystalline period ring \mathbf{B}_{cris} . The resulting period $\log_F(\hat{v})$ spans a $G_{\mathbf{Q}_p} \times \langle \varphi \rangle$ -stable \mathbf{Q}_p -line on which $G_{\mathbf{Q}_p}$ acts through χ_F .

The element $\hat{v} \in \mathbf{B}_{cris}$ is transcendental over \mathbf{Z}_p and therefore property (B) implies that the Lubin-Tate endomorphism \mathfrak{f} can be recovered from the action of Frobenius on \hat{v} . Inspired by this observation, we construct in this note an appropriate substitute for \hat{v} which is directly accessible from the commuting pair (f, u) . The argument is as follows. In section 2.1, we recall some preliminary facts concerning the commutative height one p -adic dynamical system (f, u) . In particular, we recall the *logarithm*, \log_f , of the commuting pair. Next, in section 2.2, we attach to (f, u) a character $\chi_f : G_K \rightarrow u'(0)^{\mathbf{Z}_p} \subseteq \mathbf{Z}_p^\times$, where K is some particular finite extension of \mathbf{Q}_p . The character χ_f is (*a posteriori*) a restriction of the character attached to the Tate module of the formal group for which f and u are endomorphisms. Then, in section 3, we show the character χ_f is crystalline of weight one. This is achieved by constructing an element $x_0 \in \tilde{\mathbf{A}}^+$ such that $[\chi_f(g)]_f(x_0) = g(x_0)$, where $[\chi_f(g)]_f$ is the unique \mathbf{Z}_p -iterate of u with linear term $\chi_f(g)$. We show the logarithm converges in \mathbf{A}_{cris} when evaluated at x_0 and the resulting period $t_f := \log_f(x_0)$ generates a G_K -stable \mathbf{Q}_p -line V_f of exact filtration one. Multiplicity one guarantees V_f is stable under crystalline Frobenius. Taking π_f to denote the crystalline Frobenius eigenvalue on V_f , we show $[\pi_f]_f$, the multiplication by π_f endomorphism of F_u , converges when evaluated at x_0 and satisfies $[\pi_f]_f(x_0) = \varphi(x_0)$. The proof is concluded by showing $[\pi_f]_f$ is a Lubin-Tate endomorphism of F_u and therefore F_u is defined over \mathbf{Z}_p by (4).

Remark 1.1. There is recent work by Berger [Ber14a] which uses methods similar to ours to study certain iterate extensions. Berger's work is related to the phenomena which occur when the hypotheses of Lubin and Tate's result (4) are weakened and one assumes merely that \mathfrak{f} is a lift of Frobenius i.e. $\mathfrak{f}'(0)$ is not assumed to be prime. This paper can be seen as an orthogonal generalization of Lubin and Tate's hypotheses.

2. THE p -ADIC DYNAMICAL SYSTEM

2.1. The Logarithm and Roots. Throughout this paper, we will assume $f, u \in S_0(\mathbf{Z}_p)$ are a fixed pair of power series satisfying:

Assumption 2.1. *Assume:*

- $f \circ u = u \circ f$,
- u is invertible and has infinite order, and
- $v_p(\mathfrak{f}'(0)) = 1$ and f has exactly p roots in $\mathfrak{m}_{\mathbf{C}_p}$.

The goal of this section is to remind the reader of some of the properties of the following two structures associated by Lubin (in [Lub94]) to the commuting pair (f, u) . The first is the set of f -preiterates of 0:

$$\Lambda_f := \{\pi \in \mathfrak{m}_{\mathbf{C}_p} : f^{\circ n}(\pi) = 0 \text{ for some } n \in \mathbf{Z}_+\}.$$

The second is the *logarithm* of f ; this is the unique series $\log_f \in S_0(\mathbf{Q}_p)$ such that

$$\log'_f(0) = 1 \text{ and } \log_f(f(x)) = f'(0) \log_f(x).$$

In the context that f and u are endomorphisms of a formal group F/\mathbf{Z}_p , the set Λ_f is the set of p^∞ -torsion points of F and \log_f is the logarithm of F .

One essential tool for understanding a p -adic power series and its roots is its Newton polygon. Given a series,

$$g(x) := \sum_{i=0}^{\infty} a_i x^i \in \mathbf{C}_p[[x]],$$

the Newton polygon $\mathcal{N}(g)$ of g is the convex hull in the (v, w) -Cartesian plane, \mathbf{R}^2 , of the set of vertical rays extending upwards from the points $(i, v_p(a_i))$. Let $pr_1 : \mathbf{R}^2 \rightarrow \mathbf{R}$ denote the projection map to the first coordinate. The boundary of $\mathcal{N}(g)$ is the image of an almost everywhere defined piecewise linear function $B_g : pr_1(\mathcal{N}(g)) \rightarrow \mathbf{R}^2$. The derivative of B_g is almost everywhere defined and increasing. The points of the boundary of $\mathcal{N}(g)$ where the slope jumps are called the vertices of $\mathcal{N}(g)$, whereas the maximal connected components of the boundary of $\mathcal{N}(g)$ where the slope is constant are called the segments. The width of a segment is the length of its image under pr_1 . The following data can be read off from the Newton polygon of g :

Theorem 2.2. *The radius of the maximal open disk in \mathbf{C}_p centered at 0 on which g converges is equal to $\lim_{v \rightarrow \infty} p^{\frac{dB_g}{dv}}$.*

Theorem 2.3. *The series g has a root in its maximal open disk of convergence of valuation λ if and only if the Newton polygon of g has a segment of slope $-\lambda$ of finite, positive width. The width of this slope is equal to the number of roots of g (counting multiplicity) of valuation λ .*

Theorem 2.4 (Weierstrass Preparation Theorem). *If the series g is defined over a complete extension L/\mathbf{Q}_p contained in \mathbf{C}_p and the Newton polygon of g has a segment of slope $-\lambda$ of finite, positive width then $g = g_1 g_2$ where g_1 is a monic polynomial over L and $g_2 \in L[[x]]$, such that the roots of g_1 are exactly the roots of g (with equal multiplicity) of valuation λ . We call g_1 the Weierstrass polynomial corresponding to the slope $-\lambda$.*

For proofs of these statements we refer the reader to [Kob77].

In this paper, we will be concerned with roots of power series in $\mathfrak{m}_{\mathbf{C}_p}$ and therefore focus on segments of Newton polygons of negative slope.

Proposition 2.5. *The endpoints of the segments of the Newton polygon of $f^{\circ n}$ which have negative slope are $(p^k, n - k)$ for each integer k with $0 \leq k \leq n$.*

Proof. We prove the claim by induction on n . We begin by proving the claim in the case that $n = 1$. Because $f(0) = 0$ and $f'(0) \neq 0$, the Newton polygon of f has a vertex at $(1, v_p(f'(0))) = (1, 1)$. Since f was assumed to have p roots in $\mathfrak{m}_{\mathbf{C}_p}$, the initial finite slope of $\mathcal{N}(f)$ must be negative. Furthermore, because f is defined over \mathbf{Z}_p , the Newton polygon

of f is contained in the quadrant $\mathbf{R}_{\geq 0}^2$ and all vertices have integer coordinates. Therefore, the end of the initial segment of $\mathcal{N}(f)$ must have a vertex on the v -axis. By the Weierstrass Preparation Theorem, the nonzero roots of f corresponding to this segment satisfy an Eisenstein polynomial and as such they are simple. Since f has exactly p roots in $\mathfrak{m}_{\mathbf{C}_p}$, the Newton polygon of f has a vertex at $(p, 0)$ and all nonzero roots in $\mathfrak{m}_{\mathbf{C}_p}$ are accounted for by the initial finite segment of $\mathcal{N}(f)$.

Let $n \geq 1$. The base case of our inductive argument already tells us a lot about $f^{\circ n+1}$. For example because $\mathcal{N}(f)$ has $(p, 0)$ has a vertex on the v -axis,

$$f(x) \equiv a_p x^p \pmod{(p, x^{p+1})},$$

where $a_p \in \mathbf{F}_p^\times$. It follows

$$f^{\circ n+1}(x) \equiv a_p^{n+1} x^{p^{n+1}} \pmod{(p, x^{p^{n+1}+1})}.$$

Therefore, $\mathcal{N}(f^{\circ n+1})$ has a vertex at $(p^{n+1}, 0)$. Similarly, because $\mathcal{N}(f)$ has a vertex at $(1, 1)$ and f has a fixed point at 0, the Newton polygon $\mathcal{N}(f^{\circ n+1})$ has a vertex at $(1, n+1)$. The v coordinate of a vertex abutting any finite, negatively sloped segment of $\mathcal{N}(f^{\circ n+1})$ must occur between 1 and p^{n+1} .

Now assume we have shown the claim for n . The roots of $f^{\circ n+1}$ in $\mathfrak{m}_{\mathbf{C}_p}$ are either roots of $f^{\circ n}$ or roots of $f^{\circ n}(x) - \pi$ where π is one of the $p-1$ nonzero roots of f . For any nonzero root π of f , the Newton polygon of $f^{\circ n}(x) - \pi$ consists a single segment of slope $-1/(p^{n+1} - p^n)$ and width p^n . Thus $\mathcal{N}(f^{\circ n+1})$ contains a segment of slope $-1/(p^{n+1} - p^n)$. This slope is shallower than the slope of any segment of $\mathcal{N}(f^{\circ n})$ and therefore the segment of $\mathcal{N}(f^{\circ n+1})$ of slope $-1/(p^{n+1} - p^n)$ must be the final negatively sloped segment. For each nonzero root π of f , the Weierstrass polynomial corresponding to this single negatively sloped segment of $\mathcal{N}(f^{\circ n}(x) - \pi)$ is Eisenstein over $\mathbf{Z}_p[\pi]$ and therefore its roots are distinct. Because the nonzero roots of f are distinct, there are exactly $p^n(p-1)$ roots of $f^{\circ n+1}$ of valuation $-1/(p^{n+1} - p^n)$ and therefore $\mathcal{N}(f^{\circ n+1})$ has a vertex at $(1, p^n)$.

The remaining negative segments of $\mathcal{N}(f^{\circ n+1})$ must correspond to roots of $f^{\circ n+1}$ which are roots of $f^{\circ n}$. By the inductive hypothesis, one can deduce the segments of $\mathcal{N}(f^{\circ n})$ of finite, negative slope have width $p^k - p^{k-1}$ and slope $-1/(p^k - p^{k-1})$ where k runs over the integers $1 \leq k \leq n$. Hence, all Weierstrass polynomials of $f^{\circ n}$ are Eisenstein over \mathbf{Z}_p and therefore all roots of $f^{\circ n}$ in $\mathfrak{m}_{\mathbf{C}_p}$ are simple. It follows, since every root of $f^{\circ n}$ in $\mathfrak{m}_{\mathbf{C}_p}$ is a root of $f^{\circ n+1}$, the Newton polygon $\mathcal{N}(f^{\circ n+1})$ contains for each segment of $\mathcal{N}(f^{\circ n})$ a segment of the same slope and at minimum the same width. These segments span between the vertices $(1, n+1)$ and $(1, p^n)$. The only way for this to occur is if the boundary of the Newton polygon of $\mathcal{N}(f^{\circ n+1})$ in this range is equal to the boundary Newton polygon of $\mathcal{N}(f^{\circ n})$ translated up by one in the positive w direction. The claim follows by induction. \square

Proposition 2.5 gives us a good understanding of the position of Λ_f in the p -adic unit disc. Unpacking the data contained in the Newton polygon of $f^{\circ n}$ for each n , we see that Λ_f is the disjoint union of the sets

$$C_n := \left\{ \pi \in \Lambda_f : v_p(\pi) = \frac{1}{p^n - p^{n-1}} \right\}$$

for each $n \geq 1$ and $C_0 := \{0\}$. For $n \geq 1$, the set C_n has cardinality $p^n - p^{n-1}$ and its elements are the roots of a single Eisenstein polynomial over \mathbf{Z}_p . Each of these are expected properties of the p^∞ -torsion of a height one formal group over \mathbf{Z}_p . In that case, the set C_n is the set

of elements of exact order p^n . The same holds in the absence of an apparent formal group. Comparing the Newton polygons of f^{on} and f^{on+1} , we note C_n for $n \geq 1$ can alternatively be described as:

$$C_n = \{x \in \mathfrak{m}_{\mathbf{C}_p} : f^{on}(x) = 0 \text{ but } f^{o(n-1)}(x) \neq 0\}.$$

Because f and u commute, the closed subgroup of $S_0(\mathbf{Z}_p)$ topologically generated by u acts on Λ_f and preserves the subsets C_n . Simultaneously, because f and u are defined over \mathbf{Z}_p , the Galois group $G_{\mathbf{Q}_p}$, the absolute Galois group of \mathbf{Q}_p , acts on Λ_f , preserves the subsets C_n , and commutes with the action of $u^{o\mathbf{Z}_p}$. In the next section, we will explore these commuting actions in more detail.

Next, we recall the definition of the *logarithm* of the commuting pair (f, u) . Given any series $g \in S_0(\mathbf{Z}_p)$ such that $g'(0)$ is nonzero nor equal to a root of unity, there exists a unique series $\log_g \in S_0(\mathbf{Q}_p)$ such that,

- (1) $\log'_g(0) = 1$, and
- (2) $\log_g(g(x)) = g'(0) \log_g(x)$ [Lub94, Proposition 1.2].

Lubin shows that two series $g_1, g_2 \in S_0(\mathbf{Z}_p)$ such that $g'_1(0)$ and $g'_2(0)$ satisfy the above condition have equal logarithms, if and only if, they commute under composition [Lub94, Proposition 1.3].

Consider the sequence of series in $\mathbf{Q}_p[[x]]$ whose n -th term is:

$$\frac{f^{on}(x)}{f'(0)^n}.$$

Lubin shows that this sequence converges coefficient-wise to a series \log_f satisfying:

- (1) $\log'_f(0) = 1$
- (2) $\log_f(f(x)) = f'(0) \log_f(x)$ [Lub94, Proposition 2.2].³

The limit is the logarithm of (f, u) . It is the unique series over \mathbf{Q}_p with these properties [Lub94, Proposition 1.2].

From proposition 2.5, one obtains that the vertices of Newton polygon of \log_f are $(p^k, -k)$ where k ranges over the nonnegative integers. Therefore by Theorem 2.2, \log_f converges on $\mathfrak{m}_{\mathbf{C}_p}$. Furthermore, the Newton polygon of \log_f displays that \log_f has simple roots (as each of its Weierstrass polynomials is Eisenstein over \mathbf{Z}_p) and the root set of \log_f and Λ_f have the same number of points on any circle in $\mathfrak{m}_{\mathbf{C}_p}$. One guesses that the root set of \log_f (counting multiplicity) is Λ_f . This is true and proved by Lubin [Lub94, Proposition 2.2].

The logarithm of f is invertible in $S_0(\mathbf{Q}_p)$. Conjugation by \log_f defines a continuous, isomorphism from the centralizer of f in $S_0(\mathbf{Q}_p)$ to $\text{End}_{\mathbf{Q}_p}(\widehat{\mathbb{G}}_a)$. This map sends the power series e to $e'(0)x$. If $a \in \mathbf{Z}_p$, we denote by $[a]_f(X) := \log_f^{-1}(a(\log_f(X)))$. The series $[a]_f(X)$ is the unique element of $S_0(\mathbf{Q}_p)$, which commutes with f and acts on the tangent space at 0 by a .

Consider the subgroup $u^{o\mathbf{Z}_p} \subseteq S_0(\mathbf{Z}_p)$. By the previous paragraph, conjugation by \log_f identifies this group with the closed subgroup $u'(0)^{\mathbf{Z}_p}$ of \mathbf{Z}_p^\times . By assumption 2.1, this group is infinite. Replacing u with an appropriate finite iterate, we may assume without loss of generality that:

³One can place a topology on $\mathbf{Z}_p[[X]]$ induced by a set of valuations arising from Newton copolygons. This topology is strictly finer than the topology of coefficient-wise convergence. Lubin proves that the limit defining \log_f converges in the closure of $\mathbf{Z}_p[[X]]$ under this topology.

Assumption 2.6. *The group $u^{\circ\mathbf{Z}_p}$ is topologically isomorphic to \mathbf{Z}_p and*

$$v_p(1 - u'(0)^m) = v_p(1 - u'(0)) + v_p(m)$$

for all $m \in \mathbf{Z}_p$.

2.2. A Character Arising from a Commuting Pair. Let F be a height one formal group over \mathbf{Z}_p . A fundamental invariant attached to F is its Tate module, $T_p F$. Given that F is height one, $T_p F$ is a rank one \mathbf{Z}_p -module on which $G_{\mathbf{Q}_p}$ acts through a character $\chi_F : G_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_p^\times$. This map is surjective.

Simultaneously, the automorphism group, $\text{Aut}_{\mathbf{Z}_p}(F)$, acts on $T_p F$ and commutes with the action of $G_{\mathbf{Q}_p}$. This action is through the character which sends an automorphism to multiplication by its derivative at 0. Because the automorphism group $\text{Aut}_{\mathbf{Z}_p}(F)$ is isomorphic to \mathbf{Z}_p^\times via this character, given any $g \in G_{\mathbf{Q}_p}$ there is a unique $s_g \in \text{Aut}_{\mathbf{Z}_p}(F)$ such that for all nonzero $v \in T_p F$ the equality $g.v = s_g.v$ holds. The value $\chi_F(g)$ can be recovered from s_g as $\chi_F(g) = s'_g(0)$.

This fact should be considered remarkable for it allows one to recover the character χ_F without any explicit knowledge of the additive structure of $T_p F$. Rather χ_F can be completely deduced from the structure of the $G_{\mathbf{Q}_p}$ -orbit of any nonzero element v as a $G_{\mathbf{Q}_p} \times \text{Aut}_{\mathbf{Z}_p}(G)$ -set.

In this section, guided by this observation, we will attach to our commuting pair (f, u) a character χ_f from the Galois group of a certain finite extension K of \mathbf{Q}_p to \mathbf{Z}_p^\times . The character χ_f arises from the Tate module of the latent formal group for which f and u are endomorphisms.

We begin by defining sequences of elements of $\mathfrak{m}_{\mathbf{C}_p}$ which will substitute for the rôle of elements of the Tate module of F .

Definition 2.7. [Lub94, p. 329] *An f -consistent sequence is a sequence of elements (s_1, s_2, s_3, \dots) of $\mathfrak{m}_{\mathbf{C}_p}$ such that $f(s_1) = 0$ and for $i > 1$, $f(s_i) = s_{i-1}$.*

Denote the set of all f -consistent sequences whose first entry is nonzero by \mathcal{T}_0 .

Proposition 2.8. *The Galois group $G_{\mathbf{Q}_p}$ acts transitively on \mathcal{T}_0 .*

Proof. The n -th coordinate of any f -consistent sequence in \mathcal{T}_0 is a root of $f^{\circ n}$ which is not a root of $f^{\circ n-1}$. It is enough to show that $G_{\mathbf{Q}_p}$ acts transitively on these elements. Comparing the Newton polygons of $f^{\circ n}$ and $f^{\circ n-1}$, we deduce that the set of such elements satisfy a degree $p^n - p^{n-1}$ Eisenstein polynomial over \mathbf{Q}_p . This polynomial is irreducible, and hence $G_{\mathbf{Q}_p}$ acts transitively on its roots. \square

Choose an f -consistent sequence $\Pi = (\pi_1, \pi_2, \pi_3, \dots)$ such that $\pi_1 \neq 0$. The sequence Π will be fixed throughout the remainder of this paper. Because $u \in S_0(\mathbf{Z}_p)$ is nontrivial, it can have only finitely many fixed points in $\mathfrak{m}_{\mathbf{C}_p}$. Let k be the largest integer such that $u(\pi_k) = \pi_k$. As $u^{\circ\mathbf{Z}_p} \cong \mathbf{Z}_p$ is a p -group and there are $p - 1$ nonzero roots of f , the integer k is at least 1. Let \mathcal{T}_{π_k} denote the set of all f -consistent sequences (s_1, s_2, s_3, \dots) such that $s_i = \pi_i$ for $i \leq k$.

Proposition 2.9. *\mathcal{T}_{π_k} is a torsor for the closed subgroup of $S_0(\mathbf{Z}_p)$ generated by u .*

Proof. The subgroup of $S_0(\mathbf{Z}_p)$ topologically generated by u is isomorphic to \mathbf{Z}_p . Therefore, as \mathcal{T}_{π_k} is infinite, if $u^{\circ\mathbf{Z}_p}$ acts transitively on \mathcal{T}_{π_k} it must also act freely. To prove this action is transitive, we invoke the counting powers of the orbit stabilizer theorem.

We begin by calculating the fixed points of iterates of u . Consider the set

$$\Lambda_u := \{\pi \in \mathfrak{m}_{\mathbf{C}_p} : u^{\circ n}(\pi) = \pi \text{ for some } n \in \mathbf{Z}_+\}$$

of u -periodic points in $\mathfrak{m}_{\mathbf{C}_p}$. Lubin proves $\Lambda_u = \Lambda_f$ [Lub94, Proposition 4.2.1]. Let e be a finite iterate of u . Consider the series $e(x) - x$. The roots of this series are the fixed points of e , and hence each root is an element of Λ_u . Since e is defined over \mathbf{Z}_p , the set of roots of e in $\mathfrak{m}_{\mathbf{C}_p}$ is a finite union of $G_{\mathbf{Q}_p}$ -orbits in $\Lambda_u = \Lambda_f$. The $G_{\mathbf{Q}_p}$ -orbits in Λ_f are the collection of sets

$$C_n = \{x \in \mathfrak{m}_{\mathbf{C}_p} : f^{\circ n}(x) = 0 \text{ but } f^{\circ n-1}(x) \neq 0\},$$

where n ranges over the positive integers and $C_0 = \{0\}$. Evaluation under f is an e -equivariant surjection from C_n to C_{n-1} . Hence the fixed points of e are contained in $\bigcup_{i \leq n} C_i$ for some n . By [Lub94, Proposition 4.5.2] and [Lub94, Proposition 4.3.1], respectively, the roots of $e(x) - x$ are simple and $e(x) - x \not\equiv 0 \pmod{p}$. Thus, the negative slopes of the Newton polygon of $e(x) - x$ must span between $(1, v_p(1 - e'(0)))$ and the v -axis. Since when $n \geq 1$ the set C_n is the full set of roots of an Eisenstein polynomial \mathbf{Z}_p , each of the sets C_n account for a segment in the Newton polygon of $e(x) - x$ with a decline of one unit. We conclude that the roots of $e(x) - x$ and hence the fixed points of e are simple and equal to $\bigcup_{i \leq v_p(1 - e'(0))} C_i$.

From this calculation, we deduce that $v_p(1 - u'(0)) = k$, and the point-wise stabilizer of $\pi \in C_n$ is the set

$$\{e \in u^{\circ \mathbf{Z}_p} : v_p(1 - e'(0)) \geq n\}.$$

By assumption 2.6, it follows the stabilizer of any $\pi \in C_n$ for $n \geq k$ is the group $u^{\circ p^{n-k}} \mathbf{Z}_p$.

We now show that $u^{\circ \mathbf{Z}_p}$ acts transitively on \mathcal{T}_{π_k} . As in the proof of proposition 2.8, it is enough to show that $u^{\circ \mathbf{Z}_p}$ acts transitively on the set of possible values for the n -th coordinate of a sequence in \mathcal{T}_{π_k} . Let $n \geq k$ and consider the set

$$W_n(\pi_k) = \{\pi \in \mathfrak{m}_{\mathbf{C}_p} : f^{\circ n-k}(\pi) = \pi_k\}.$$

The set $W_n(\pi_k)$ are the possible values of an n -th coordinate of a sequence in \mathcal{T}_{π_k} . Because $\pi_k \neq 0$, the Newton polygon of $f^{\circ n-k} - \pi_k$ has exactly one segment of positive slope. The length of this segment is p^{n-k} and its slope is $\frac{1}{p^{n-1}(p-1)}$. Hence, $W_n(\pi_k)$ has order p^{n-k} and is contained in C_n . It follows that the $u^{\circ \mathbf{Z}_p}$ -orbit of a point $\pi \in W_n(\pi_k)$ has order

$$|u^{\circ \mathbf{Z}_p} / u^{\circ p^{n-k}} \mathbf{Z}_p| = p^{n-k} = |W_n(\pi_k)|.$$

We conclude the action of $u^{\circ \mathbf{Z}_p}$ on $W_n(\pi_k)$ is transitive. \square

Let $K := \mathbf{Q}_p(\pi_k)$. The transitive $G_{\mathbf{Q}_p}$ action on \mathcal{T}_0 restricts to a transitive G_K -action on \mathcal{T}_{π_k} . The fixed field of the kernel of this action is $K_\infty := \mathbf{Q}_p(\pi_i | i > 0)$. Let $\sigma \in G_K$. Because \mathcal{T}_{π_k} is a torsor for $u^{\circ \mathbf{Z}_p}$ there is a unique element $u_\sigma \in u^{\circ \mathbf{Z}_p}$ such that $u_\sigma \Pi = \sigma \Pi$. We set $\chi_f(\sigma) := u'_\sigma(0)$. The power series u_σ can be recovered from $\chi_f(\sigma)$ as $u_\sigma = [\chi_f(\sigma)]_f$.

Proposition 2.10. *The map $[\chi_f]_f : G_K \rightarrow u^{\circ \mathbf{Z}_p}$ is a surjective group homomorphism satisfying $[\chi_f(\sigma)]_f(\pi_i) = \sigma(\pi_i)$ for all $i > 0$. The fixed field of the kernel of $[\chi_f]_f$ is K_∞ .*

Proof. Outside of the fact that $[\chi_f]_f$ is a homomorphism, this proposition follows directly from the discussion above. We show $[\chi_f]_f$ is a homomorphism. Let $\sigma_1, \sigma_2 \in G_K$. Then

$$\begin{aligned} [\chi_f(\sigma_2)]_f[\chi_f(\sigma_1)]_f(\Pi) &= [\chi_f(\sigma_1)]_f[\chi_f(\sigma_2)]_f(\Pi) \\ &= [\chi_f(\sigma_1)]_f\sigma_2(\Pi) \\ &= \sigma_2([\chi_f(\sigma_1)]_f\Pi) \\ &= \sigma_2\sigma_1(\Pi). \end{aligned}$$

By uniqueness, it follows $[\chi_f(\sigma_2)]_f[\chi_f(\sigma_1)]_f = [\chi_f(\sigma_2\sigma_1)]_f$ and $[\chi_f]_f$ is a homomorphism. \square

The character χ_f comes via a 'geometric' construction and therefore should have good behavior from the viewpoint of p -adic Hodge theory. In the next section we show this is the case. Specifically, we show χ_f is crystalline of weight 1.

3. THE HODGE THEORY OF χ_f

3.1. Fontaine's Period Rings. To show that χ_f is crystalline, we must show that χ_f occurs (by definition) as a G_K -sub-representation in Fontaine's period ring \mathbf{B}_{cris} . In this section, we will recall the construction of this ring as well as several other of Fontaine's rings of periods (for which the original constructions can be found in [Fon94]). We will follow the naming conventions of Berger [Ber04].

The construction of the period rings begin in characteristic p . Let $\tilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbf{C}_p}/p$. The ring $\tilde{\mathbf{E}}^+$ inherits an action of the Galois group $G_{\mathbf{Q}_p}$ from the action of $G_{\mathbf{Q}_p}$ on $\mathcal{O}_{\mathbf{C}_p}$. Additionally, by virtue of being a ring of characteristic p , the ring $\tilde{\mathbf{E}}^+$ comes equipped with a Frobenius endomorphism $Frob_{\tilde{\mathbf{E}}^+}$. The map $Frob_{\tilde{\mathbf{E}}^+}$ commutes with the action of $G_{\mathbf{Q}_p}$.

The map $Frob_{\tilde{\mathbf{E}}^+}$ is an isomorphism and therefore $\tilde{\mathbf{E}}^+$ is perfect. Set $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$, the Witt vectors of $\tilde{\mathbf{E}}^+$. The formality of the Witt vectors implies that the commuting actions of $Frob_{\tilde{\mathbf{E}}^+}$ and $G_{\mathbf{Q}_p}$ on $\tilde{\mathbf{E}}^+$ lift, respectively, to commuting actions of $Frob_{\tilde{\mathbf{A}}^+}$ and $G_{\mathbf{Q}_p}$ on $\tilde{\mathbf{A}}^+$ as ring endomorphisms. The lift of $Frob_{\tilde{\mathbf{E}}^+}$ is denoted by φ .

Let $\tilde{\mathbf{B}}^+ := \tilde{\mathbf{A}}^+[\frac{1}{p}]$. Fontaine constructs a surjective $G_{\mathbf{Q}_p}$ -equivariant ring homomorphism $\theta : \tilde{\mathbf{B}}^+ \rightarrow \mathbf{C}_p$ which can be defined as follows. The map θ is the unique homomorphism which is continuous with respect to the p -adic topologies on $\tilde{\mathbf{B}}^+$ and \mathbf{C}_p , and which maps the Teichmüller representative $\{x\}$ to $\lim_{n \rightarrow \infty} x_n$ where x_n is any lift of x^{1/p^n} to $\mathcal{O}_{\mathbf{C}_p}$.

The period ring \mathbf{B}_{dR}^+ is defined as the $\ker(\theta)$ -adic completion of $\tilde{\mathbf{B}}^+$. The ring \mathbf{B}_{dR}^+ is topologized so that it inherits the topology generated by the intersection of the $\ker(\theta)$ -adic and p -adic topologies of the dense subring $\tilde{\mathbf{B}}^+$. The map θ extends to a continuous $G_{\mathbf{Q}_p}$ -equivariant surjection $\theta : \mathbf{B}_{dR}^+ \rightarrow \mathbf{C}_p$ and $\ker(\theta)$ is principal. The ring \mathbf{B}_{dR}^+ is abstractly isomorphic to $\mathbf{C}_p[[t]]$. The map θ induces a \mathbf{N} -graded filtration on \mathbf{B}_{dR}^+ where $\text{Fil}^i \mathbf{B}_{dR}^+ := \ker(\theta)^i$. The de Rham ring of periods, \mathbf{B}_{dR} , is defined as the fraction field of \mathbf{B}_{dR}^+ . The filtration on \mathbf{B}_{dR}^+ extends to a \mathbf{Z} -graded filtration on \mathbf{B}_{dR} .

Let V/\mathbf{Q}_p be a finite dimensional representation of the absolute Galois group of a p -adic field E . We say V is de Rham if the E -dimension of $\mathbf{D}_{dR}^*(V) := \text{Hom}_{\mathbf{Q}_p[[G_E]]}(V, \mathbf{B}_{dR})$ is equal to the dimension of V . The filtration on \mathbf{B}_{dR} induces a filtration $\mathbf{D}_{dR}^*(V)$. The nonzero graded pieces of this filtration are called the weights of V . The weight of the cyclotomic character is 1.

The ring \mathbf{B}_{dR} does not admit a natural extension of the Frobenius endomorphism, φ , of $\tilde{\mathbf{A}}^+$. To rectify this, Fontaine defines a ring \mathbf{B}_{cris} of \mathbf{B}_{dR} on which the endomorphism φ extends. The ring \mathbf{B}_{cris} is defined as follows: Let \mathbf{A}_{cris}° be the PD-envelope of $\tilde{\mathbf{A}}^+$ with respect to the ideal $\ker(\theta) \cap \tilde{\mathbf{A}}^+$. The ring \mathbf{A}_{cris}° is a subring of \mathbf{B}_{dR}^+ . We define \mathbf{A}_{cris} to be the p -adic completion of \mathbf{A}_{cris}° . One can show that the inclusion of \mathbf{A}_{cris}° into \mathbf{B}_{dR}^+ extends naturally to an inclusion of \mathbf{A}_{cris} into \mathbf{B}_{dR}^+ . Under this inclusion \mathbf{A}_{cris} is identified with the set of elements of the form $\sum_{i=0}^{\infty} a_n \frac{t^n}{n!}$ where $a_n \in \tilde{\mathbf{A}}^+$ such that $\lim_{n \rightarrow \infty} a_n = 0$ in the p -adic topology and $t \in \ker(\theta) \cap \tilde{\mathbf{A}}^+$. The ring \mathbf{B}_{cris}^+ is defined as $\mathbf{B}_{cris}^+ := \mathbf{A}_{cris}[\frac{1}{p}]$. The ring \mathbf{B}_{cris}^+ contains a unique $G_{\mathbf{Q}_p}$ -stable \mathbf{Q}_p -line on which $G_{\mathbf{Q}_p}$ acts by the cyclotomic character. The ring \mathbf{B}_{cris} is defined from \mathbf{B}_{cris}^+ by inverting any non-zero vector in this line.

Let E_0 be the maximal unramified extension of \mathbf{Q}_p contained in E . A finite dimensional representation V/\mathbf{Q}_p of the absolute Galois group of a p -adic field E is called crystalline if the E_0 -dimension of $\mathbf{D}_{cris}^*(V) := \text{Hom}_{\mathbf{Q}_p[[G_E]]}(V, \mathbf{B}_{cris})$ is equal to the dimension of V . The endomorphism φ extends uniquely to an endomorphism of each of the rings $\mathbf{A}_{cris}^\circ, \mathbf{A}_{cris}, \mathbf{B}_{cris}^+$, and \mathbf{B}_{cris} .

3.2. The Universal f -Consistent Sequence. Let $\mathbf{Z}_p[[x_1]]$ be the ring of formal power series over \mathbf{Z}_p in the indeterminate x_1 . In this section we will define a ring, A_∞ , containing $\mathbf{Z}_p[[x_1]]$, which parameterizes f -consistent sequences in certain topological rings. The initial term of such a sequence will be parameterized by x_1 . We will show that one can define a continuous injection $A_\infty \hookrightarrow \tilde{\mathbf{A}}^+$ under which the image of A_∞ is G_K -stable and satisfies $\sigma(x_1) = [\chi_f(\sigma)]_f(x_1)$.

We begin by defining A_∞ . For each positive integer i , set $A_i := \mathbf{Z}_p[[x_i]]$, the ring of formal power series over \mathbf{Z}_p in the indeterminate x_i . Denote, for each i , the homomorphism that maps $x_i \mapsto f(x_{i+1})$ by $[f'(0)]_f^* : A_i \rightarrow A_{i+1}$. We define $A_\infty := \varinjlim A_i$ to be the colimit of the rings A_i with respect to the transition maps $[f'(0)]_f^*$. Finally, the ring A_∞ is defined to be the p -adic completion of A_∞° .

We view A_∞ as a topological ring under the adic topology induced by the ideal (p, x_1) . The reader should be aware that while A_∞ is complete with respect to the finer p -adic topology, it is not complete with respect to the (p, x_1) -adic topology. Under the topology on A_∞ the elements x_i are topologically nilpotent and together topologically generate A_∞ as a \mathbf{Z}_p -algebra.

The ring A_∞ is a natural object to consider from the viewpoint of nonarchemidian dynamics for it is universal for f -consistent sequences in the following sense: given any p -adically complete, ind-complete adic \mathbf{Z}_p -algebra S and any f -consistent sequence $s := (s_1, s_2, s_3, \dots)$ of topologically nilpotent elements in S there exists a unique homomorphism $\phi_s : A_\infty \rightarrow S$ such that $\phi_s(x_i) = s_i$. Conversely, any homomorphism from A_∞ to S arises as ϕ_s for some sequence s . In other words, A_∞ represents the functor from p -adically complete, ind-complete adic \mathbf{Z}_p -algebras to sets which sends an algebra S to the set of f -consistent sequences of topologically nilpotent elements in S .

Alternatively, we claim that A_∞ is canonically isomorphic to $W(A_\infty/pA_\infty)$, the Witt vectors of the residue ring A_∞/pA_∞ , and hence A_∞ is closely connected to constructions in p -adic Hodge theory. The fact that $A_\infty \cong W(A_\infty/pA_\infty)$ follows from the observation that A_∞ is a strict p -ring, i.e. that A_∞ is complete and Hausdorff for the p -adic topology, p is not a zero divisor in A_∞ , and A_∞/pA_∞ is perfect [Haz09]. Of these three criteria only the third is not immediately obvious for A_∞ . To see that A_∞/pA_∞ is perfect, we recall:

Theorem 3.1. [Lub94, Corollary 6.2.1, p. 343] *Let k be a finite field, and let $\mathbf{u}, \mathbf{f} \neq 0$ be invertible and non-invertible, respectively, in $S_0(k)$, commuting with each other. Then either \mathbf{u} is a torsion element of $S_0(k)$ or \mathbf{f} has the form $\mathbf{f}(x) = \mathbf{a}(x^{p^h})$ with $h \in \mathbf{Z}_+$ and $\mathbf{a} \in S_0(k)$ is invertible.*

In our case, theorem 3.1 implies $f(x) \equiv a(x^p) \pmod{p}$ for some invertible series $a \in S_0(\mathbf{F}_p)$. It follows:

$$\begin{aligned} A_\infty/pA_\infty &\cong A_\infty^\circ/pA_\infty^\circ \\ &\cong \varinjlim A_i/pA_i \\ &\cong \varinjlim_{x_i \mapsto a(x_{i+1}^p)} \mathbf{F}_p[[x_i]] \\ &\cong \varinjlim_{x_i \mapsto a(x_{i+1})^p} \mathbf{F}_p[[x_i]] \\ &\cong \varinjlim_{a^{o^i}(x_i) \mapsto (a^{o^{i+1}}(x_{i+1}))^p} \mathbf{F}_p[[a^{o^i}(x_i)]] \\ &\cong \varinjlim_{y_i \mapsto y_{i+1}^p} \mathbf{F}_p[[y_i]], \end{aligned}$$

where $y_i := a^{o^i}(x_i)$. As this final ring is perfect the claim follows.

Our goal is to find an injection $A_\infty \hookrightarrow \tilde{\mathbf{A}}^+$ with particularly nice properties. Since A_∞ is a strict p -ring, any injection $A_\infty/pA_\infty \hookrightarrow \tilde{\mathbf{E}}^+$ lifts canonically to an injection $A_\infty \hookrightarrow \tilde{\mathbf{A}}^+$. Let $\Pi := (\pi_1, \pi_2, \pi_3, \dots)$ be the f -consistent sequence of elements in $\mathfrak{m}_{\mathbf{C}_p}$ fixed in section 2.2. Then, as $f(x) \equiv a(x^p) \pmod{p}$, we observe for each $i \in \mathbf{Z}_+$ the series $\hat{\pi}_i := \langle a^{o^{k-i}}(\pi_k) \pmod{p} \rangle_{k \geq i} \in \tilde{\mathbf{E}}^+$. Define $\hat{\Pi} := (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \dots)$. The sequence $\hat{\Pi}$ is f -consistent and consists of topologically nilpotent elements of $\tilde{\mathbf{E}}^+$. Let $\phi_{\hat{\Pi}} : A_\infty \rightarrow \tilde{\mathbf{E}}^+$ be the homomorphism associated to $\hat{\Pi}$.

Proposition 3.2. *The kernel of the map $\phi_{\hat{\Pi}} : A_\infty \rightarrow \tilde{\mathbf{E}}^+$ is pA_∞ . The induced injection $\phi_{\hat{\Pi}} : A_\infty/pA_\infty \rightarrow \tilde{\mathbf{E}}^+$ identifies A_∞/pA_∞ with a G_K -stable subring of $\tilde{\mathbf{E}}^+$ such that for all $\sigma \in G_K$ and positive integers i the equality $\sigma(\phi_{\hat{\Pi}}(x_i)) = \phi_{\hat{\Pi}}([\chi_f(\sigma)]_f(x_i))$ holds.*

Proof. The ring $\tilde{\mathbf{E}}^+$ has characteristic p and therefore $\phi_{\hat{\Pi}}$ factors through

$$A_\infty/pA_\infty \cong A_\infty^\circ/pA_\infty^\circ \cong \varinjlim A_i/pA_i.$$

To prove the proposition, we show that the restriction of the map induced by $\phi_{\hat{\Pi}}$ to A_i/pA_i is an injection and satisfies $\sigma(\phi_{\hat{\Pi}}(x_i)) = \phi_{\hat{\Pi}}([\chi_f(\sigma)]_f(x_i))$ for all $\sigma \in G_K$.

The ring A_i/pA_i is isomorphic to $\mathbf{F}_p[[x_i]]$. Therefore, any homomorphism out of A_i/pA_i is either injective or has finite image. We claim the latter is false for the map induced by $\phi_{\hat{\Pi}}$. To see this, consider $S_0(\mathbf{F}_p)$ acting on $\phi_{\hat{\Pi}}(A_i/pA_i)$. We claim that the stabilizer of $\phi_{\hat{\Pi}}(x_i)$ is trivial and hence $\phi_{\hat{\Pi}}(x_i)$ has infinite orbit. Let $z \in S_0(\mathbf{F}_p)$ be a nontrivial element and $n = \deg(z(x) - x)$. Then

$$z(\phi_{\hat{\Pi}}(x_i)) - x_i = \langle z(a^{o^{k-i}}(\pi_k)) - \pi_k \pmod{p} \rangle_{k \geq i}.$$

For any positive integer k , the coordinate $z(a^{o^{k-i}}(\pi_k)) - \pi_k \pmod{p}$ is the image under the reduction map $\mathcal{O}_{\mathbf{C}_p} \rightarrow \mathcal{O}_{\mathbf{C}_p}/p$ of an element of p -adic valuation $\frac{n}{p^k - p^{k-1}}$. Hence for $k \gg 0$, we have $z(a^{o^{k-i}}(\pi_k)) - \pi_k \pmod{p}$ is nonzero. It follows z does not stabilize $\phi_{\hat{\Pi}}(x_i)$.

The proof that G_K acts in the desired way on $\phi_{\hat{\Pi}}(x_i)$ is by direct calculation. Let $\sigma \in G_K$. Then

$$\begin{aligned}
\sigma(\phi_{\hat{\Pi}}(x_i)) &= \sigma(\hat{\pi}_i) \\
&= \langle \sigma(a^{\circ k-i}(\pi_k)) \pmod{p} \rangle_{k \geq i} \\
&= \langle a^{\circ k-i}(\sigma(\pi_k)) \pmod{p} \rangle_{k \geq i} \\
&= \langle a^{\circ k-i}([\chi_f(\sigma)]_f(\pi_i)) \pmod{p} \rangle_{k \geq i} \\
&= \langle [\chi_f(\sigma)]_f(a^{\circ k-i}(\pi_i)) \pmod{p} \rangle_{k \geq i} \\
&= [\chi_f(\sigma)]_f(\hat{\pi}) \\
&= \phi_{\hat{\Pi}}([\chi_f(\sigma)]_f(x_i)).
\end{aligned}$$

□

Because A_∞ is a strict p -ring, the injection induced by $\phi_{\hat{\Pi}} : A_\infty/pA_\infty \hookrightarrow \tilde{\mathbf{E}}^+$ lifts canonically to an injection $W(\phi_{\hat{\Pi}}) : A_\infty \hookrightarrow \tilde{\mathbf{A}}^+$. Henceforth, we will identify A_∞ with its image under $W(\phi_{\hat{\Pi}})$. The G_K action on A_∞/pA_∞ lifts functorially to a G_K action on A_∞ ; furthermore, the map $W(\phi_{\hat{\Pi}})$ is G_K -equivariant and identifies A_∞ with a G_K -stable subring of $\tilde{\mathbf{A}}^+$. Our next goal is to precisely describe the action of G_K on A_∞ . The following theorem provides the rigidity needed to understand these lifts.

Let \mathbf{F}_q be a finite extension of \mathbf{F}_p and consider an invertible series $\omega \in S_0(\mathbf{F}_q)$ such that $\omega'(0) = 1$. The absolute ramification index of ω is defined to be the limit

$$e(\omega) := \lim_{n \rightarrow \infty} (p-1)v_x(\omega^{\circ p^n}(x) - x)/p^{n+1}.$$

Theorem 3.3. [LMS02, Proposition 5.5]. *Assume $e(\omega) < \infty$, then the separable normalizer of $\omega^{\circ \mathbf{Z}_p}$ in $S_0(\mathbf{F}_q)$ is a finite extension $\omega^{\circ \mathbf{Z}_p}$ by a group of order dividing $e(\omega)$.*

Lemma 3.4. *There exists a positive integer m such that $a^{\circ N} \in u^{\circ \mathbf{Z}_p} \pmod{p}$.*

Proof. Note as f and u commute, so too do a and the reduction of u modulo p . One observes, upon considering the Newton polygon $u^{\circ n}(x) - x$, that $v_x(u^{\circ n}(x) - x) = |1 - u'(0)^n|_p$. By assumption 2.6, the absolute ramification index of u is therefore finite. The result follows by Theorem 3.3. □

Proposition 3.5. *Let $\sigma \in G_K$. Then $\sigma(x_i) = [\chi_f(\sigma)]_f(x_i)$.*

Proof. Consider the sequence

$$[\chi_f(\sigma)]_f(\Pi^{\text{univ}}) := ([\chi_f(\sigma)]_f(x_i))_{i \in \mathbf{Z}^+}$$

of elements in A_∞ . Because f and $[\chi_f(\sigma)]_f$ commute, the sequence $[\chi_f(\sigma)]_f(\Pi^{\text{univ}})$ is f -consistent. Hence, by the universal property of A_∞ , there exists an endomorphism $[\chi_f(\sigma)]_f^* := \phi_{[\chi_f(\sigma)]_f(\Pi^{\text{univ}})}$ of A_∞ which maps x_i to $[\chi_f(\sigma)]_f(x_i)$.

We wish to show that the Witt lift $W(\sigma)$ is equal to $[\chi_f(\sigma)]_f^*$. The former map is defined by its action mod p on Teichmüller representatives. Denote the Teichmüller mapping by $\{*\} : A_\infty/pA_\infty \rightarrow A_\infty$. For $\bar{\alpha} \in A_\infty/pA_\infty$ the element $\{\bar{\alpha}\}$ can be defined as follows: let α_n be any lift of $\bar{\alpha}^{1/p^n}$, then $\{\bar{\alpha}\} := \lim_{n \rightarrow \infty} \alpha_n^{p^n}$. The map $\{*\}$ is well defined as this limit converges and is independent of the choice of lifts α_n . The automorphism $W(\sigma)$ is the unique map such that $W(\sigma)\{\bar{\alpha}\} = \{\sigma(\bar{\alpha})\}$.

We claim the set of Teichmüller lifts of $x_i \pmod p$ together topologically generate A_∞ (in the (p, x_1) -adic topology) as a \mathbf{Z}_p -algebra. To see this, let A_∞^{tm} be the sub-algebra topologically generated by these elements. We show $A_\infty^{tm} = A_\infty$. First note that the residue rings $A_\infty^{tm}/pA_\infty^{tm}$ and A_∞/pA_∞ are equal. Next observe that A_∞^{tm} is a strict p -ring. To see this, note A_∞^{tm} is closed in the p -adic topology on A_∞ and is therefore p -adically complete and Hausdorff, A_∞^{tm} is a sub-algebra of A_∞ and hence p is not a zero divisor in A_∞^{tm} , and $A_\infty^{tm}/pA_\infty^{tm} = A_\infty/pA_\infty$ and hence the residue ring $A_\infty^{tm}/pA_\infty^{tm}$ is perfect. It follows the image of $A_\infty^{tm}/pA_\infty^{tm} = A_\infty/pA_\infty$ under the Teichmüller map lies in A_∞^{tm} . But A_∞ (in the p -adic topology) is topologically generated as a \mathbf{Z}_p -algebra by the image of A_∞/pA_∞ under the Teichmüller map. We conclude $A_\infty^{tm} = A_\infty$.

Therefore, to show $W(\sigma) = [\chi_f(\sigma)]_f^*$, it is enough to show $W(\sigma(\{x_i\})) := \{\sigma(x_i)\}$ is equal to $[\chi_f(\sigma)]_f^*(\{x_i\})$ for all $i \in \mathbf{Z}^+$. Fix a positive integer i . By lemma 3.4, there exists an integer $N > 0$ and a p -adic number k such that $a^{\circ N} = u^{\circ k}$. Hence, as $f(x) \equiv a(x^p) \pmod p$, it follows for every positive integer m the element $u^{\circ -km}(x_{Nm+i})$ is a lift of $x_i^{p^{1/Nm}} \pmod p$. Similarly, $u^{\circ -km}([\chi_f(\sigma)]_f(x_{Nm+i}))$ is a lift of $(\sigma(x_i))^{p^{1/Nm}} \pmod p$. Using these lifts we will compare the action of $W(\sigma)$ and $[\chi_f(\sigma)]_f^*$ on $\{x_i\}$. Observe

$$\begin{aligned} [\chi_f(\sigma)]_f^*(\{x_i\}) &= \lim_{m \rightarrow \infty} ([\chi_f(\sigma)]_f^*(u^{\circ -km}(x_{Nm+i})))^{p^{Nm}} \\ &= \lim_{m \rightarrow \infty} (u^{\circ -km}([\chi_f(\sigma)]_f(x_{Nm+i})))^{p^{Nm}} \\ &= \lim_{m \rightarrow \infty} (u^{\circ -km}([\chi_f(\sigma)]_f(x_{Nm+i})))^{p^{Nm}} \\ &= W(\sigma)(\{x_i\}). \end{aligned}$$

□

Remark 3.6. The ring A_∞ is not an unfamiliar object in the study of formal groups (or more generally p -divisible groups). If f is the endomorphism of a formal group $F := \mathrm{Spf}(\mathbf{Z}_p[[x_1]])$, the completion of A_∞ is the ring of global functions on the universal cover of F (see [SW13, Section 3.1]).

3.3. The p -adic Regularity of χ_f . In this section, we will examine the image of the sequence $\langle x_i \rangle$ under \log_f . To ensure convergence, we will appeal to the following lemma of Berger:

Lemma 3.7. [Ber14b, Lemma 3.2] *Let E be a finite extension of \mathbf{Q}_p and take $L(X) \in E[[X]]$. If $x \in \mathbf{B}_{dR}^+$, then the series $L(x)$ converges in \mathbf{B}_{dR}^+ if and only if $L(\theta(x))$ converges in \mathbf{C}_p .*

By construction, the image of A_∞ lies in $\tilde{\mathbf{A}}^+$ and therefore $\theta(x_i) \in \mathcal{O}_{\mathbf{C}_p}$. The map from A_∞ to $\tilde{\mathbf{A}}^+$ is a lift of the map $\phi_{\tilde{\mathbf{A}}} : A_\infty \rightarrow \tilde{\mathbf{E}}^+$ which sends x_i to $\hat{\pi}_i := \langle a^{\circ k-i}(\pi_k) \pmod p \rangle_{k \geq i}$. Therefore, $\theta(x_i) \equiv \pi_i \pmod p$ and hence $\theta(x_i) \in \mathfrak{m}_{\mathbf{C}_p}$. Now the series $\log_f(X)$ converges on $\mathfrak{m}_{\mathbf{C}_p}$ and so we see from lemma 3.7 that $\log_f(x_i)$ converges in \mathbf{B}_{dR}^+ for all $i \in \mathbf{Z}^+$.

Set $x_0 := f(x_1)$. Then $\theta(x_0) \in \mathfrak{m}_{\mathbf{C}_p}$ and hence $\log_f(x_0)$ converges in \mathbf{B}_{dR}^+ . We define $t_f := \log_f(x_0)$. We note that

$$(f'(0))^{n+1} \log_f(x_n) = \log_f(f^{\circ n+1}(x_n)) = \log_f(f(x_1)) = t_f$$

and therefore the values $\log_f(x_n)$ are \mathbf{Q}_p -multiples of t_f . We call t_f a fundamental period of f . The reader should be warned that t_f depends not only on f but also our choice of f -consistent

sequence II. If f is an endomorphism of a formal group defined over \mathbf{Z}_p any two fundamental periods differ by a p -adic unit.

How does G_K act on t_f ? Let $\sigma \in G_K$, then

$$\begin{aligned}\sigma(t_f) &= \sigma \log_f(f(x_1)) \\ &= \log_f(f(\sigma(x_1))) \\ &= \log_f(f \circ [\chi_f(\sigma)]_f(x_1)) \\ &= f'(0)\chi_f(\sigma) \log_f(x_1) \\ &= \chi_f(\sigma)t_f\end{aligned}$$

Therefore, assuming $t_f \neq 0$, the period t_f generates a G_K -stable \mathbf{Q}_p -line of \mathbf{B}_{dR}^+ on which G_K acts through χ_f . From this it follows χ_f is de Rham of some positive weight. Our first proposition of this section shows that this is in fact the case.

Proposition 3.8. *χ_f is de Rham of weight 1.*

Proof. The claim will follow if we can show $t_f \in \text{Fil}^1 \mathbf{B}_{dR}^+ \setminus \text{Fil}^2 \mathbf{B}_{dR}^+$.

We begin by showing $t_f \in \text{Fil}^1 \mathbf{B}_{dR}^+$. Assume this were not the case. Then χ_f would be Rham of weight 0. However, all such representations are potentially unramified and K_∞ , the fixed field of $\ker \chi_f$, is an infinitely ramified \mathbf{Z}_p -extension of K . It follows that $t_f \in \text{Fil}^1 \mathbf{B}_{dR}^+$.

Next we show $t_f \notin \text{Fil}^2 \mathbf{B}_{dR}^+$. Since $0 = \theta(t_f) = \log_f(\theta(x_0))$, we observe $\theta(x_0)$ is a root of \log_f . The quotient $\mathbf{B}_{dR}^+/\text{Fil}^2 \mathbf{B}_{dR}^+$ is a square zero extension of $\mathbf{B}_{dR}^+/\text{Fil}^1 \mathbf{B}_{dR}^+ \cong \mathbf{C}_p$. Since all roots of \log_f in $\mathfrak{m}_{\mathbf{C}_p}$ are simple, we conclude $t_f \notin \text{Fil}^2 \mathbf{B}_{dR}^+$. \square

Lemma 3.9. *The derivative of the logarithm, \log'_f , is an element of $\mathbf{Z}_p[[x]]$.*

Proof. Since the series \log_f converges on $\mathfrak{m}_{\mathbf{C}_p}$, so too does \log'_f . The Newton polygon of \log'_f contains the point $(0, 0)$. Hence, $\log'_f \in \mathbf{Z}_p[[x]]$ if and only if \log'_f has no roots in $\mathfrak{m}_{\mathbf{C}_p}$. Assume for the sake of contradiction \log'_f had a root π in $\mathfrak{m}_{\mathbf{C}_p}$. Then

$$(3.1) \quad \log'_f(u(\pi))u'(\pi) = (\log_f \circ u)'(\pi) = u'(0) \log'_f(\pi) = 0.$$

The power series $u'(x)$ is invertible in $\mathbf{Z}_p[[x]]$, so equation 3.1 implies $\log'_f(u(\pi)) = 0$. Hence, $u \circ \mathbf{Z}_p$ acts on the set of roots of \log'_f . Since u preserves valuation and \log'_f has only finitely many roots of any fixed valuation, some power of u fixes π . From the equality $\Lambda_u = \Lambda_f$, we conclude the element π is a root of \log_f . But this is a contradiction as \log_f has simple roots. \square

Proposition 3.10. *The period t_f is an element of \mathbf{A}_{cris} , and χ_f is crystalline.*

Proof. By lemma 3.9:

$$\log_f(x) = \sum_{i=1}^{\infty} \frac{a_n}{n} x^n$$

where $a_n \in \mathbf{Z}_p$ for all $n \in \mathbf{Z}^+$. Therefore,

$$t_f = \sum_{i=1}^{\infty} a_n (n-1)! \frac{x_0^n}{n!}.$$

It follows if $\theta(x_0) = 0$, then $t_f \in \mathbf{A}_{cris}$. To see $\theta(x_0) = 0$, note that $\theta(x_0)$ is a root of \log_f in $\mathfrak{m}_{\mathbf{C}_p}$ and

$$\theta(x_0) \equiv \theta(f(x_1)) \equiv f(\pi_1) \equiv 0 \pmod{p}.$$

As 0 is the unique root of \log_f of p -adic valuation greater than 1, the element $\theta(x_0) = 0$ and the claim follows. \square

4. THE MAIN THEOREM

4.1. Constructing the Formal Group. Let V_f be the \mathbf{Q}_p -line of \mathbf{B}_{cris}^+ generated by the period t_f . In section 3.3, we deduced that V_f is a G_K -sub-representation of \mathbf{B}_{cris}^+ isomorphic to χ_f . Since K/\mathbf{Q}_p is totally ramified, any G_K -representation on a vector space V/\mathbf{Q}_p has multiplicity in \mathbf{B}_{cris}^+ at most one. Therefore, V_f is the unique \mathbf{Q}_p -line of \mathbf{B}_{cris}^+ isomorphic to χ_f . It follows V_f is preserved by crystalline Frobenius. Let $\pi_f \in \mathbf{Q}_p$ be the eigenvalue of crystalline Frobenius acting on V_f . Because V_f has weight one, the p -adic valuation of π_f is equal to 1.

The next lemma is fundamental. We will show that the equality $\varphi(\log_f(x_0)) = \pi_f \log_f(x_0)$ implies that $\varphi(x_0) = [\pi_f]_f(x_0)$. This will be enough to show $[\pi_f]_f \in S_0(\mathbf{Z}_p)$ and satisfies $[\pi_f]_f(x) \equiv x^p \pmod{p}$, from which we will deduce that f and u are endomorphisms of an integral formal group by a lemma of Lubin and Tate.

Lemma 4.1. *The power series $[\pi_f]_f \in \pi_f x + x^2 \mathbf{Z}_p[[x]]$ and satisfies $[\pi_f]_f(x) \equiv x^p \pmod{p}$.*

Proof. First note that by construction $x_0 \in \tilde{\mathbf{A}}^+ \subseteq \mathbf{B}_{cris}^+$ and so x_0 is acted upon by crystalline Frobenius. Consider $\varphi(x_0)$. We claim $\theta(\varphi(x_0)) = 0$. To see this, observe

$$\log_f(\theta(\varphi(x_0))) = \theta(\varphi(\log_f(x_0))) = \theta(\pi_f t_f) = 0.$$

Hence, $\theta(\varphi(x_0))$ is a root of $\log_f(x)$. As $x_0 \in \tilde{\mathbf{A}}^+$ and $\theta(x_0) = 0$, the image of crystalline Frobenius satisfies $\theta(\varphi(x_0)) \in p\mathcal{O}_{\mathbf{C}_p}$. The unique root of $\log_f(x)$ in $p\mathcal{O}_{\mathbf{C}_p}$ is 0, therefore we conclude $\theta(\varphi(x_0)) = 0$.

Because $\theta(x_0) = 0$ and $\theta(\varphi(x_0)) = 0$, any series $e(x) \in \mathbf{Q}_p[[x]]$ converges in \mathbf{B}_{dR}^+ when evaluated at x_0 or $\varphi(x_0)$. From this we deduce the equality

$$\log_f(\varphi(x_0)) = \varphi(t_f) = \pi_f t_f = \log_f([\pi_f]_f(x_0)).$$

Additionally, we may apply \log_f^{-1} to both sides, obtain convergent results and deduce $\varphi(x_0) = [\pi_f]_f(x_0)$.

Write

$$[\pi_f]_f(x) = \pi_f x + \sum_{n=2}^{\infty} a_n x^n,$$

where $a_n \in \mathbf{Q}_p$. We prove by induction on n that $a_n \in \mathbf{Z}_p$. Let $n \geq 2$ and assume $a_i \in \mathbf{Z}_p$ for all $i < n$. Set

$$[\pi_f]_f^{<n} := \pi_f x + \sum_{i=2}^{n-1} a_i x^i.$$

Observe the equality

$$[\pi_f]_f^{<n}(x_0) \equiv \varphi(x_0) \equiv x_0^p \pmod{\text{Fil}^n \mathbf{B}_{dR}^+ \cap \tilde{\mathbf{A}}^+ + p\tilde{\mathbf{A}}^+}.$$

Hence, as the homomorphism $\mathbf{Z}_p[[x]] \rightarrow \tilde{\mathbf{A}}^+$ given by mapping $x \mapsto x_0$ is injective and (x) is the pullback of $Fil^n \mathbf{B}_{dR}^+ \cap \tilde{\mathbf{A}}^+$, it must be the case that $[\pi_f]_f^{<n}(x) \equiv x^p \pmod{p}$. Express $f(x) = f'(0)x + f^{\geq 2}(x)$ where $f^{\geq 2}(x) \in x^2 \mathbf{Z}_p[[x]]$. Note that

$$f \circ [\pi_f]_f(x) \equiv f([\pi_f]_f^{<n}(x)) + a_n f'(0)x^n \pmod{x^{n+1}}$$

and

$$[\pi_f]_f(f(x)) \equiv [\pi_f]_f^{<n}(f(x)) + a_n f'(0)^n x^n \pmod{x^{n+1}}.$$

As $f(x)$ and $[\pi_f]_f(x)$ commute,

$$(4.1) \quad f([\pi_f]_f^{<n}(x)) - [\pi_f]_f^{<n}(f(x)) \equiv a_n(f'(0) - f'(0)^n)x^n \pmod{x^{n+1}}.$$

The left hand side of (4.1) is a power series over \mathbf{Z}_p and reduces modulo p to

$$f([\pi_f]_f^{<n}(x)) - [\pi_f]_f^{<n}(f(x)) \equiv f(x^p) - f(x)^p \equiv 0 \pmod{p}.$$

Therefore, the right hand side of (4.1) is a power series over $p\mathbf{Z}_p$. Since $v_p(f'(0) - f'(0)^n) = 1$, it follows $a_n \in \mathbf{Z}_p$. By induction we conclude $[\pi_f]_f$ is an element of $\pi_f x + x^2 \mathbf{Z}_p[[x]]$. From the equality $\varphi(x_0) = [\pi_f]_f(x_0)$, we observe $[\pi_f]_f(x) \equiv x^p \pmod{p}$. \square

Theorem 4.2. *Let f and u be a commuting pair of elements in $S_0(\mathbf{Z}_p)$. If*

- $f'(0)$ is prime in \mathbf{Z}_p and f has exactly p roots in $\mathfrak{m}_{\mathbf{C}_p}$,
- u is invertible and has infinite order,

then there exists a unique formal group law F/\mathbf{Z}_p such that $f, u \in \text{End}_{\mathbf{Z}_p}(F)$. The formal group law F is isomorphic to $\widehat{\mathbb{G}}_m$ over the ring of integers of the maximal unramified extension of \mathbf{Q}_p .

Proof. By [LT65, Theorem 1.1], the series $[\pi_f]_f$ is an endomorphism of a unique height one formal group over F/\mathbf{Z}_p . Furthermore, f and u are endomorphisms of F . The second half of the claim follows as all height one formal groups over \mathbf{Z}_p are forms of $\widehat{\mathbb{G}}_m$ and are trivialized over the ring of integers of the maximal unramified extension of \mathbf{Q}_p . \square

5. ACKNOWLEDGEMENTS

I would like to thank Frank Calegari, Vlad Serban, Paul VanKoughnett, and the anonymous referee for their helpful edits of earlier drafts of this paper.

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