THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
2 Homework - FALL SESSION 2007
110.617 - Number Theory

1. Let K be a field, $L = K(X)$ the field of rational functions in an indeterminate over K, $P(X)$ an irreducible polynomial of $K[X]$, and Φ_P the valuation on L associated with P.
   Determine the valuation ring, the valuation ideal and the residue (class) field of Φ_P.
   Describe the completion of L for the valuation Φ_P, when $P = X$.
   When is the completion of L locally compact for a $P(X)$-adic valuation?

2. We say that we have a *periodic Hensel expansion* for a number $\alpha \in \mathbb{Q}$ if there exist two integers $n_0 \in \mathbb{Z}$ and $r > 0$ such that
   $$\alpha_{n+r} = \alpha_n, \qquad n > n_0,$$
   where
   $$\alpha = \alpha_m p^m + \alpha_{m+1} p^{m+1} + \cdots + \alpha_n p^n + \cdots \qquad (m \in \mathbb{Z}).$$
   Show that the Hensel expansion of $\alpha \in \mathbb{Q}_p$ is periodic if and only if $\alpha \in \mathbb{Q}$.

3. In $\mathbb{Q}_3$ determine the first five terms of the Hensel expansion of
   $$\ln 4 = \ln(1+3) = 3 - \frac{3^2}{2} + \frac{3^3}{3} - \frac{3^4}{4} + \frac{3^5}{5} \cdots$$

4. Show that for $p \neq 2$, 1 is the only p-th root of unity in $\mathbb{Q}_p$.

5. Let F be a complete valuation field for a discrete valuation Φ of the valuation ring $\mathcal{O}$. Let $f(x)$ be a monic polynomial in $\mathcal{O}[x]$ such that there exist $\alpha_1 \in \mathcal{O}$ with $|f(\alpha_1)| < 1$ and $|f'(\alpha_1)| = 1$. Show that the sequence
   $$\alpha_1,\ \alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)},\ \cdots, \alpha_n = \alpha_{n-1} - \frac{f(\alpha_{n-1})}{f'(\alpha_{n-1})}$$
   converges in $\mathcal{O}$ to a root of f.

6. Show that the polynomial $X^2 + X + \alpha$, where $\alpha \in \mathbb{Z}$, has a solution in $\mathbb{Q}_2$ if and only if α is even.

   Hint: Apply the previous exercise to deduce the "only if" part.