2 HMW SOLUTIONS

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1. The valuation ring A of $L = K(X)$ for the P-adic valuation is the set of rational functions $\frac{R}{S}$ st $(R,S)=1$ and $(S,P)=1$

The valuation ideal I is the set of rational functions $\frac{R}{S}$ st $(S,R)=1$, $(S,P)=1$ and $P \mid R$

The residue field is A/I : $A/I \cong \frac{A \cap K[X]}{I \cap K[X]} \cong$
$\cong \frac{K[X]}{P(X)K[X]}$

If $P = X$, the completion of L is the set of Laurent series with coefficients in K having only finitely many negative indices.

If P is an irreducible polynomial, the completion of L is the set of Laurent series in $P(X)$ with coefficients in a complete system of representatives of $K[X]/P(X)K[X]$ (hence is a finite extension of K) and having only finitely many negative indices.

For the completion of $K(X)$ to be locally compact (for a $P(X)$-adic valuation) it is necessary and sufficient that ϕ_P is discrete (which is the case here) and that the residue field is a finite extension of K. Therefore it is necessary and sufficient that K is a finite field.

2. Let $\alpha \in \mathbb{Q}_p$, $\alpha = \alpha_m p^m + \dots$ and let assume that the Hensel expansion of α is periodic.

Then, $\exists$ n_0 and r st: $\alpha_{n+r} = \alpha_n$ $n \geq n_0$.

Therefore: $\alpha = \alpha_m p^m + \dots + \alpha_{n_0-1} p^{n_0-1} + \alpha_{n_0}(p^{n_0} + p^{n_0+r} + \dots) + \alpha_{n_0+1}(p^{n_0+1} + p^{n_0+r+1} + \dots) + \dots + \alpha_{n_0+r-1}(p^{n_0+r-1} + p^{n_0+2r-1} + \dots) =$

$$= \alpha_m p^m + \dots + \alpha_{n_0-1} p^{n_0-1} + \frac{\alpha_{n_0} p^{n_0} + \dots + \alpha_{n_0+r-1} p^{n_0+r-1}}{1-p^r}$$

Hence $\alpha \in \mathbb{Q}$.

Conversely, if $\alpha \in \mathbb{Q}$, then $\alpha = \frac{m}{n}$ $m, n \in \mathbb{Z}$

Then exists n' st if $n > 0$, then $n'n = (p^r - 1)p^s$ (application of Lagrange's Theorem)

Let assume that $m > 0$ and $n < 0$. Hence $\exists$ $n' > 0$ st $\alpha = \frac{mn'}{p^s(1-p^r)}$ By possibly modifying n' w.m.a

$$\alpha = \frac{mn'}{p^s(1-p^s)} = \frac{mn'}{1-p^s} + \frac{mn'}{p^s}$$

$$\rightarrow \alpha = M + \frac{R}{1-p^s} + \frac{mn'}{p^s} \qquad 0 \leq R < p^s - 1, \ M \in \mathbb{N}$$

If we set: $M = m_0 + m_1 p + \dots + m_i p^i$

$mn' = m_0' + m_1' p + \dots + m_j' p^j$

$R = r_0 + r_1 p + \dots + r_k p^k$ $\quad k < s$ for $l+1$ a multiple of s and $l \geq \sup(i, j)$, we have

$\alpha = \alpha_t p^t + \dots + \alpha_l p^l + r_0 p^{l+1} + r_1 p^{l+2} + \dots + r_k p^{l+k+1} + r_0 p^{s+l+1} + \dots$

The Hensel expansion of α is therefore periodic with period s for $n \geq l+1$.

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3. Want to find $\alpha \in \mathbb{N}$ st:

$$|\alpha - \log 4| \leq \left(\tfrac{1}{3}\right)^5 \quad \text{and} \quad \alpha < 3^5$$

$$\left| \log 4 - \left(3 - \frac{3^2}{2} + \frac{3^3}{3} - \frac{3^4}{4}\right)\right| < \left(\tfrac{1}{3}\right)^5, \quad \text{since}$$

$$\left|\frac{3^n}{n}\right| < \left(\tfrac{1}{3}\right)^{5^2} \quad \text{whenever } n > 5$$

Therefore:

$$3 - \frac{3^2}{2} + \frac{3^3}{3} - \frac{3^4}{4} = 12 - \frac{99}{4} = 12 + \frac{243 - 99}{4} \bmod 3^5$$

$$= 48 \bmod 3^5 = 1 \cdot 3 + 2 \cdot 3^2 + 3^3 \bmod 3^5$$

4. Let $\xi \neq 1$ be a p-th root of 1 in $\mathbb{Q}_p$.
Set $\xi - 1 = u$. Hence u is a root of the equation:

$$p + u\binom{p}{2} + u^2\binom{p}{3} + \dots + u^{p-2}\binom{p}{p-1} + u^{p-1} = 0$$

$$\left|\binom{p}{2}\right| = \left|\binom{p}{3}\right| = \dots = \left|\binom{p}{p-1}\right| = \frac{1}{p} \longrightarrow$$

$$|u^{p-1}| = \frac{1}{p} \quad (\text{by ultrametric inequality})$$

and therefore

$$|u| = \left(\frac{1}{p}\right)^{\frac{1}{p-1}} \quad \text{impossible if } u \in \mathbb{Q}_p$$

5. $(\alpha_2 - \alpha_1) f'(\alpha_1) = -f(\alpha_1)$

By Taylor's formula, with $n = \deg(f)$, we have

$$f(\alpha_2) = (\alpha_2 - \alpha_1)^2 \left[\frac{f''(\alpha_1)}{2!} + \dots + \frac{(\alpha_2 - \alpha_1)^{n-2}}{n!} f^{(n)}(\alpha_1)\right].$$

$$\frac{f^{(k)}(X)}{k!} \in \mathcal{O}[X] \to \left|\frac{f^{(k)}(\alpha_1)}{k!}\right| \leq 1 \quad \text{hence}$$

$$|f(\alpha_2)| \leq |\alpha_2 - \alpha_1|^2 = |f(\alpha_1)|^2 \quad \text{and}$$

$$|f(\alpha_2)| \le |\alpha_2 - \alpha_1|^2 = |f(\alpha_1)|^2$$ and

$$f'(\alpha_2) = f'(\alpha_1) + (\alpha_2 - \alpha_1) f''(\alpha_1) + \dots$$

$$|\alpha_2 - \alpha_1| < 1 \to |f'(\alpha_2)| = |f'(\alpha_1)| = 1$$

Then, by induction we show that:

$$|f'(\alpha_i)| = 1, \quad |f(\alpha_i)| < |f(\alpha_1)|^i$$ and that

$$|\alpha_i - \alpha_{i-1}| < |f(\alpha_1)|^{i-1}$$

Therefore $\lim_{i \to \infty} |\alpha_i - \alpha_{i-1}| = 0$

The sequence $(\alpha_i)_i$ is therefore Cauchy in F so it has a limit α:

$$\alpha = \alpha - \frac{f(\alpha)}{f'(\alpha)} \to f(\alpha) = 0$$ because $|f'(\alpha)| = 1$

6. If $X^2 + X + a$ has a root in $\mathbb{Q}_2$ this root must belong to $\mathbb{Z}_2$ as the polynomial is monic and has coefficients in $\mathbb{Z}$.

Let α be a root, $\alpha \in \mathbb{Z}_2 \to |\alpha| \le 1$.

If $|\alpha| < 1$, then the equality

$$a = -\alpha - \alpha^2 \to |a| < 1,$$ therefore a is even

If $|\alpha| = 1$, then $\alpha = 1 + 2\alpha_1$ $(|\alpha_1| \le 1)$ and therefore

$$a = -1 - 2\alpha_1 - (1 + 2\alpha_1)^2 = -4\alpha_1^2 - 6\alpha_1 - 2 \equiv 0 \bmod 2$$

Conversely:

if a is even, the image of the polynomial $X^2 + X + a$ in $(\mathbb{Z}/2\mathbb{Z})[X] = \mathbb{F}_2[X]$ is the polynomial $X^2 + X$, which splits in the polynomial X^2+X, which splits in $\mathbb{F}_2[X]$ into a product of 2 monic, rel. prime factors.

Therefore by Hensel's lemma the polynomial X^2+X+a is reducible in $\mathbb{Q}_2[X]$