1 Introduction

Let $X$ be a curve over a finite extension $K$ of $\mathbb{Q}_p$, which is $k$-split degenerate, for $k$ the residue field. It is well known that such curve admits a $p$-adic uniformization by a $p$-adic Schottky group $\Gamma$ acting on the Bruhat-Tits tree $\Delta_K$. We associate $C^*$-algebras to certain subgraphs $\Delta$ of the Bruhat-Tits tree and construct corresponding dynamical cohomologies $\mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma)$ that resemble the construction at arithmetic infinity given in [6]. We introduce a Dirac operator $D$, which depends on the graded structure of the dynamical cohomology, in such a way that the data

$$\left( C^*(\Delta/\Gamma), \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma) \oplus \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma), D \right)$$

(1.1)

give a spectral triple in the sense of Connes.

We recover, from a certain family of zeta functions associated to the spectral triple, the local Euler factor $L_p(H^1(X), s)$ of the curve $X$, in the form of a regularized determinant as computed by Deninger. The advantage of this construction is that it provides, in the case we are considering, a natural geometric interpretation of the infinite-dimensional cohomology theory introduced by Deninger, in terms of dynamics of walks on the graph $\Delta/\Gamma$.

We propose a way of extending the construction to the case where the curve is not $k$-split degenerate by enlarging the dual graph of the special fiber by new edges, in such a way that we also obtain embeddings in the dynamical cohomology of the first cohomology group of the components of the special fiber.
2 Directed graphs

We begin by recalling some generalities about graphs that we use throughout the paper.

A directed graph $E$ consists of data $E = (E^0, E^1, r, s, t)$, where $E^0$ is the set of vertices, $E^1$ is the set of oriented edges $w = (e, c)$, where $e$ is an edge of the graph and $c = \pm 1$ is a choice of orientation. The set $E^1_+$ consists of a choice of orientation for each edge, namely one element in each pair $(e, \pm 1)$. The maps $r, s : E^1 \to E^0$ are the range and source maps, and $i$ is the involution on $E^1$ defined by $(w) = (e, -c)$.

A morphism $f$ of directed graphs $E$ and $\bar{E}$ consists of maps $f^0 : E^0 \to \bar{E}^0$ and $f^1 : E^1 \to \bar{E}^1$ which satisfy $f^i \circ \tau = \bar{\tau} \circ f^i$, $f^i \circ s = \bar{s} \circ f^i$, and $f^i \circ t = \bar{t} \circ f^i$, for $i = 0, 1$. The morphism $f$ is a monomorphism if the $f^i$ are invertible, for $i = 0, 1$. This defines the automorphism group of a directed graph, which we denote $\text{Aut}(E)$. A morphism $f$ of directed graphs is a covering map if the $f^i$ are onto, for $i = 0, 1$, and $f^1$ gives a bijection $f^1 : s^{-1}(\bar{v}) \sim \bar{s}^{-1}(f(v))$ and the same with respect to the range map.

A directed graph is finite if $E^0$ and $E^1$ are finite sets. It is row finite if at each vertex $v \in E^0$ there are at most finitely many edges $w$ in $E^1_+$ such that $s(w) = v$. The graph is locally finite if each vertex emits and receives at most finitely many oriented edges in $E^1$. A vertex $v$ in a directed graph is a sink if there is no edge in $E^1_+$ with source $v$. We denote by $\sigma(E)$ the subset of $E^0$ given by the sinks.

A juxtaposition of oriented edges $w_1 \ldots w_k$ is said to be admissible if $w_2 \neq t(w_1)$ and $r(w_1) = s(w_2)$. A (finite, infinite, doubly infinite) walk in a directed graph $E$ is an admissible (finite, infinite, doubly infinite) sequence of elements in $E^1$. A (finite, infinite, doubly infinite) path in $E$ is a walk where all edges in the sequence are in $E^1_+$. We denote by $W^n(E)$ the set of walks of length $n$, by $W^*(E) = \cup_n W^n(E)$, by $W^+_*(E)$ the set of infinite walks, and by $W(E)$ the set of doubly infinite walks. Similarly, we introduce the analogous notation $\mathcal{P}^n(E)$, $\mathcal{P}^*_*(E)$, and $\mathcal{P}_+^*(E)$ for paths. We will drop the explicit mention of the graph $E$ in the notation for walks and paths, when no confusion arises. We denote by $\sigma^*(E) \subset \mathcal{P}^*_*(E)$ the set of paths that end at a sink. A directed graph is a directed tree if, for any two vertices, there exists a unique walk in $W^*_*(E)$ connecting them.

The universal cover $\Delta$ of a connected directed graph $E$, endowed with a choice of a base point $v_0 \in E^0$, is a directed tree obtained by setting $\Delta^0 = W^*_*(v_0)$, the set of all walks in $E$ that start at $v_0$, $\Delta^1 = \{(w, w) \in W^*_*(v_0) \times E^1 : r(w) = s(w), r(w, w) = \omega w$ and $(\omega w, t(w)) \}$, where $\omega$ is a choice of orientation. The fundamental group of $E$, with respect to the choice of base point $v_0$, is given by $\Gamma = \{\gamma \in W^*_*(v_0) : r(\gamma) = v_0\}$.

Let $G$ be a subgroup of $\text{Aut}(E)$. We can then form the quotient $E/G$, which is also a directed graph. In particular, if $\Delta$ is the universal cover of a directed graph $E$ and $\Gamma$ is the fundamental group, with respect to the choice of a base point $v_0 \in E^0$, then we have an isomorphism of directed graphs $E \simeq \Delta/\Gamma$. The map $\Delta \to E$ is a covering map.
Two paths \( \omega \) and \( \tilde{\omega} \) in \( \mathcal{P}^+ \) are \textit{shift-tail equivalent}, if there exists an \( N \geq 1 \) and a \( k \in \mathbb{Z} \) such that \( \omega_i = \tilde{\omega}_{i-k} \) for all \( i \geq N \). The boundary of a directed tree \( \Delta \) is given by \( \partial \Delta = (\mathcal{P}^+ \cup \sigma^*)/\sim \), where the shift-tail equivalence is extended to the set of paths ending at a sink by the condition \( \omega \sim \tilde{\omega} \), for \( \omega, \tilde{\omega} \in \sigma^* \), if and only if \( \tau(\omega) = \tau(\tilde{\omega}) \). This definition extends to a functorial notion of boundary of directed graphs, as shown in [21].

3 Schottky-Mumford curves

Throughout this section, we denote by \( p \) a fixed prime number and by \( \mathbb{Q}_p \) the field of \( p \)-adic numbers. The field \( K \) will be a given finite extension of \( \mathbb{Q}_p \), with \( \mathcal{O} \subset K \) its ring of integers, \( m \subset \mathcal{O} \) the maximal ideal, and \( \pi \in m \) a uniformizer (i.e., \( m = (\pi) \)). Finally, we denote by \( k \) the residue field \( k = \mathcal{O}/m \). This is a finite field of cardinality \( q = \text{card}(\mathcal{O}/m) \).

Let \( V \) be a two-dimensional vector space over \( K \). We write \( \mathbb{P}^1(K) \) for the set of \( K \)-rational points of \( \mathbb{P}^1 \), the projective line over \( K \). The space \( \mathbb{P}^1(K) \) is identified with the set of \( K \)-lines passing through the origin in \( V \). Let \( G = \text{PGL}(2, K) \) be the group of automorphisms of \( \mathbb{P}^1(K) \).

In this first section, we collect some well-known facts and properties on the tree of the group \( \text{PGL}(2, K) \) and on the action of a Schottky group on such tree. Detailed explanations for the statements we recall here without proofs are contained in [12, 18].

3.1 The tree of the group \( \text{PGL}(2, K) \)

The description of the vertices of the tree associated to \( \text{PGL}(2, K) \) is as follows. One considers the set of free \( \mathcal{O} \)-modules of rank 2: \( M \subset V \). Two such modules are \textit{equivalent} \( M_1 \sim M_2 \) if there exists an element \( \lambda \in K^* \), such that \( M_1 = \lambda M_2 \). The group \( \text{GL}(V) \) of linear automorphisms of \( V \) operates on the set of such modules \textit{on the left}: \( gM = \{gm \mid m \in M\} \), \( g \in \text{GL}(V) \). Notice that the relation \( M_1 \sim M_2 \) is equivalent to the condition that \( M_1 \) and \( M_2 \) belong to the same orbit of the center \( K^* \subset \text{GL}(V) \). Hence, the group \( G = \text{GL}(V)/K^* \) operates (on the left) on the set of classes of equivalent modules.

We denote by \( \Delta^\mathcal{O}_K \) the set of such classes and by \( \{M\} \) the class of the module \( M \). Because \( \mathcal{O} \) is a principal ideals domain and every module \( M \) has two generators, it follows that

\[
\{M_1\}, \{M_2\} \in \Delta^\mathcal{O}_K, \quad M_1 \supset M_2 \Rightarrow M_1/M_2 \cong \mathcal{O}/m^l \oplus \mathcal{O}/m^k, \quad l, k \in \mathbb{N}.
\] (3.1)

The multiplication of \( M_1 \) and \( M_2 \) by elements of \( K \) preserves the inclusion \( M_1 \supset M_2 \), hence the natural number
is well defined.

Definition 3.1. The graph $\Delta_K$ of the group $\text{PGL}(2, K)$ is the infinite graph with set of vertices $\Delta^0_K$, in which two vertices $\{M_1\}$ and $\{M_2\}$ are adjacent and hence connected by an edge if and only if $d(\{M_1\}, \{M_2\}) = 1$.

The following properties characterize completely $\Delta_K$ and are well known (cf., e.g., [12, 18]).

Proposition 3.2. (1) The graph $\Delta_K$ is a connected, locally finite tree with $q + 1$ edges departing from each of its vertices.

(2) The shortest walk in $\Delta_K$ connecting two vertices $\{M_1\}$ and $\{M_2\}$ of $\Delta^0_K$ without backtracking has length $d(\{M_1\}, \{M_2\})$.

(3) The group $G$ acts (on the left) transitively on $\Delta^0_K$ and it preserves the metric $d$. \[ \square \]

The tree $\Delta_K$ is called the Bruhat-Tits tree associated to the group $G = \text{PGL}(2, K)$.

A half line in $\Delta_K$ is an infinite sequence $(\{M_n\})_{n \in \mathbb{N}}$ of vertices of $\Delta_K$ without repetitions such that $\{M_n\}$ is adjacent to $\{M_{n-1}\}$ for all $n$. Thus, a half line is given by a sequence $M_0 \supset M_1 \supset \cdots$ of free $\mathcal{O}$-modules where $M_0/M_n \cong \mathcal{O}/(\pi^n)$ for all $n$.

The subspace $K(\cap_{n \in \mathbb{N}} M_n) \subset V$ defines a point of $\mathbb{P}^1(K)$. Conversely, given a point of $\mathbb{P}^1(K)$ represented by a vector $v_1 \in V$, choose a second vector $v_2 \in V$ such that $\{v_1, v_2\}$ form a basis of $V$. Let $M_n$ be the free $\mathcal{O}$-module with basis $\{v_1, \pi^n v_2\}$. Then, $K(\cap_{n \in \mathbb{N}} M_n)$ defines the point of $\mathbb{P}^1(K)$ we started with.

Two half lines are said to be equivalent if they differ only by a finite number of vertices. An equivalence class of half lines is called an end of $\Delta_K$. The set of ends of $\Delta_K$ will be denoted by $\partial \Delta_K$ (the “boundary” of $\Delta_K$). We give $\Delta_K$ the structure of a directed graph, in such a way that this notion of boundary agrees with the one described in the previous section.

It is immediate to verify that the construction described above defines a one-to-one correspondence $\partial \Delta_K \cong \mathbb{P}^1(K)$ between equivalence classes of half lines and elements of $\mathbb{P}^1(K)$.

3.2 The action of a Schottky group on the tree $\Delta_K$

A Schottky group $\Gamma$ is a subgroup of $\text{PGL}(2, K)$ which is finitely generated and whose elements $\gamma \neq 1$ are hyperbolic (i.e., the eigenvalues of $\gamma$ in $K$ have different valuation). The
Let $\Gamma$ be a Schottky group contained in $\text{PGL}(2, K)$, which is discrete, torsion-free, and acts freely on the tree $\Gamma$. Furthermore, one can show that $\Gamma$ acts discretely at some point $z \in \mathbb{P}^1(K)$. The closure of the set of points in $\mathbb{P}^1(K)$ that are fixed points of some $\gamma \in \Gamma \setminus \{1\}$ is called the limit set of $\Gamma$. We have $\text{card}(\Lambda_\Gamma) < \infty$ if and only if $\Gamma = (\gamma)^\mathbb{Z}$, for some $\gamma \in \Gamma$. We denote by $\Omega_\Gamma = \Omega_\Gamma(K)$ the set of points on which $\Gamma$ acts discretely, equivalently said, $\Omega_\Gamma = \mathbb{P}^1(K) \setminus \Lambda_\Gamma$ is the set of points which are not limits of fixed points of elements of $\Gamma$. This set is called the domain of discontinuity for the Schottky group $\Gamma$.

A path in $\Delta_K$, infinite in both directions and with no backtracking, is called an axis of $\Delta_K$. Any two points $z_1, z_2 \in \mathbb{P}^1(K)$ uniquely define their connecting axis whose endpoints lie in the classes described by $z_1$ and $z_2$ in $\partial \Delta_K$.

Any hyperbolic element $\gamma \in \Gamma$ has two fixed points in $\mathbb{P}^1(K)$. The unique axis of $\Delta_K$ whose ends are fixed by $\gamma$ is called the axis of $\gamma$. The element $\gamma$ acts on its axis as a translation.

Suppose two axes are given in $\Delta_K$. Any path without back-tracking beginning on one axis and ending on the other and with no edges in common with either axis is said to be the bridge between them. A bridge may consist of a single point, else it is uniquely defined (Figure 3.1).

For any Schottky group $\Gamma \subset \text{PGL}(2, K)$, there is a smallest subtree $\Delta'_\Gamma \subset \Delta$ containing the axes of all elements of $\Gamma$. Equivalently said, $\Delta'_\Gamma$ is the maximal connected subgraph of $\Delta_K$ containing the axes of all elements of $\Gamma$ and the bridges between them.

The set of ends of $\Delta'_\Gamma$ in $\mathbb{P}^1(K)$ is $\Lambda_\Gamma$, the limit set of $\Gamma$. The group $\Gamma$ carries $\Delta'_\Gamma$ into itself so that the quotient $\Delta'_\Gamma / \Gamma$ is a finite graph.

The graph $\Delta'_\Gamma / \Gamma$ has an important geometric interpretation as the dual graph of the closed fiber of the minimal smooth model over $\mathbb{O}$ (k-split degenerate semistable
curve) of the algebraic curve $C$ holomorphically isomorphic to $X_{\Gamma} := \Omega_{\Gamma}/\Gamma$ (cf. [18, page 163]).

Furthermore, there is a smallest tree $\Delta_{\Gamma}$ on which $\Gamma$ acts, with vertices $\Delta^0_{\Gamma} \subset \Delta_{\Gamma}^0$, and such that the finite graph $\Delta_{\Gamma}/\Gamma$ is the dual graph of the specialization of $C$ over $\emptyset$. The set $\Delta^0_{\Gamma}$ is a subset of the set of vertices of $\Delta'_{\Gamma}$.

The degenerating curve $C$ describing the analytic uniformization $X_{\Gamma} \simeq \rightarrow C$ is a $k$-split degenerate, stable curve. When the genus of the fibers is at least 2—that is, when the Schottky group has at least $g \geq 2$ generators—the curves $X_{\Gamma}$ are called Schottky-Mumford curves.

### 3.3 Field extensions: functoriality

Let $L \supset K$ be a finite extension with ramification index $e_{L/K}$ and rings of integers $O_L$ and $O_K$. If $M \subset V$ is a free $O_K$-module of rank 2, then $M \otimes_{O_K} O_L \subset V \otimes_K L$ is a free $O_L$-module of the same rank. It is obvious that equivalent modules remain equivalent, hence one gets a natural embedding of the sets of vertices $\Delta^0_K \hookrightarrow \Delta^0_L$.

The isomorphism $(O_K/m^r) \otimes O_L \simeq O_L/m^{re_{L/K}}$ shows that distance would not, in general, be preserved under field extensions. To eliminate this disadvantage, one introduces on the graphs $\Delta_L$, for all extensions $L \supset K$, a $K$-normalized distance

$$d_K(M_1, M_2) = \frac{1}{e_{L/K}} d_L(M_1, M_2); \quad M_1, M_2 \subset V \otimes L. \quad (3.3)$$

This way, the embedding $\Delta^0_K \hookrightarrow \Delta^0_L$ becomes isometric. When $L \supset K$ ramifies, $e_{L/K} - 1$ new vertices appear in $\Delta^0_L$ between each couple of adjacent vertices in $\Delta^0_K$ (cf. Figure 3.2(c)). Moreover, $q^f + 1$ edges, each of which is of length $1/e_{L/K}$, for $f = (1/e_{L/K})[L : K]$, depart from each vertex in $\Delta^0_K$ (cf. Figure 3.2(b)).

Because $\text{PGL}(2, K) \subset \text{PGL}(2, L)$ and the natural embedding $\mathbb{P}^1(K) \subset \mathbb{P}^1(L)$ is compatible with a concept of $K$-direction of exiting from any vertex of $\Delta^0_K$, the construction is functorial under finite extensions and this process determines, for a fixed Schottky group $\Gamma \subset \text{PGL}(2, K)$, a projective system $\{X_{L, \Gamma} : [L : K] < \infty\}$ of Schottky-Mumford curves.

### 3.4 Edge orientation

We now show how to endow the graphs $\Delta_{\Gamma}$ and $\Delta_K$ with the structure of directed graph. The choice of a coordinate $z \in \mathbb{P}^1(K)$ determines a base point $v_0$ in $\Delta_K$. In fact, the points $\{0, 1, \infty\}$ on $\mathbb{P}^1(K)$ determine a unique crossroad: the unique vertex of $\Delta_K$ with the property
(a) The tree $\Delta_K$ for $K = Q_2$.

(b) $\Delta_L$ for field extensions with $f = 2$.

(c) $e_{L/K} = 2$.

Figure 3.2

That the walks without backtracking from $\tilde{v}_0$ to the three points on $\mathbb{P}^1(K)$ start from $\tilde{v}_0$ in three different directions (Figure 3.3).

In order to obtain a structure of directed graph on the tree $\Delta_K$, for each $z \in \mathbb{P}^1(K)$ we consider the unique infinite chain of edges without backtracking in $\Delta_K$ that has initial vertex $\tilde{v}_0$ and whose equivalence class modulo shift-tail equivalence is the point $z$. We declare such chain of edges to be a path in $\mathcal{P}^+(\Delta_K)$. This determines on $\Delta_K$ the structure of a directed graph, with $\partial\Delta_K = \mathbb{P}^1(K)$. This agrees with the boundary as defined in Section 3.1.

We assume here that the coordinate $z \in \mathbb{P}^1(K)$ is chosen in such a way that the crossroad $\tilde{v}_0$ of $\{0, 1, \infty\}$ is a vertex of $\Delta_L$. Then, by the same procedure, we can regard $\Delta_L$ as a directed graph with $\partial\Delta_L = \Lambda_L$. 
4 \(C^*\)-algebras of graphs

In this section, we recall the construction of \(C^*\)-algebras associated to locally finite directed graphs. We mostly follow \([1, 10, 11, 21]\). We refer the reader to the bibliography of the aforementioned papers for more information. For simplicity, we state the following results in the special case of locally finite directed graph, though the theory extends to more general directed graphs (cf., e.g., \([21]\)).

A Cuntz-Krieger family consists of a collection \(\{P_v\}_{v \in E^0}\) of mutually orthogonal projections and \(\{S_w\}_{w \in E^1}\) of partial isometries, satisfying the conditions: \(S_w^*S_w = P_{r(w)}\) and, for all \(v \in s(E^1)\), \(P_v = \sum_{w:t(w)=v} S_wS_w^*\).

The edge matrix \(A_+\) of a locally finite (or row finite) directed graph is an \((\#E^1_+) \times (\#E^1_+)\) (possibly infinite) matrix. The entries \(A_+(w_i, w_j)\) satisfy \(A_+(w_i, w_j) = 1\) if \(w_iw_j\) is an admissible path, and \(A_+(w_i, w_j) = 0\) otherwise. The Cuntz-Krieger elements \(\{P_v, S_w\}\) satisfy the relation \(S_w^*S_w = \sum A(w, v)S_vS_v^*\). The directed edge matrix of \(E\) is a \(\#E^1 \times \#E^1\) (possibly infinite) matrix with entries \(A(w_i, w_j) = 1\) if \(w_iw_j\) is an admissible walk and \(A(w_i, w_j) = 0\) otherwise.

There is a universal \(C^*\)-algebra, \(C^*(E)\), generated by a Cuntz-Krieger family. If \(E\) is a finite graph with no sinks, we have \(C^*(E) \simeq \mathcal{O}_{A_+}\), where \(\mathcal{O}_{A_+}\) is the Cuntz-Krieger algebra of the edge matrix \(A_+\). If the directed graph is a tree \(\Delta\), then \(C^*(\Delta)\) is an AF algebra strongly Morita equivalent to the commutative \(C^*\)-algebra \(C_0(\partial\Delta)\). A monomorphism of directed trees induces an injective \(*\)-homomorphism of the corresponding \(C^*\)-algebras.

If \(G \subset \text{Aut}(E)\) is a group acting freely on the directed graph \(E\), with quotient graph \(E/G\), then the crossed product \(C^*\)-algebra \(C^*(E) \rtimes G\) is strongly Morita equivalent to \(C^*(E/G)\). In particular, if \(\Delta\) is the universal covering tree of a directed graph \(E\) and \(\Gamma\) is the fundamental group, then the algebra \(C^*(E)\) is strongly Morita equivalent to \(C_0(\partial\Delta) \rtimes \Gamma\).

There is a gauge action of \(U(1)\) on the graph algebra \(C^*(E)\) given by \(\lambda : \{P_v, S_w\} \mapsto \{P_v, \lambda S_w\}\). A subset \(H\) of the set of vertices \(E^0\) of a directed graph is saturated hereditary if \(v \in H\) implies that, for all \(\omega \in \mathcal{P}^*(E)\) with \(s(\omega) = v\), also \(r(\omega) \in H\), and conversely, if any \(\omega \in \mathcal{P}^*(E)\) with \(s(\omega) = v\) satisfies \(r(\omega) \in H\), then also \(v \in H\). For a locally finite graph, there is a bijective correspondence between saturated hereditary subsets of \(E^0\) and gauge invariant closed ideals in \(C^*(E)\). In the case of a tree \(\Delta\), there is a bijection between saturated hereditary subsets of \(\Delta^0\) and open sets in \(\partial\Delta\). This is proved for the more general (not necessarily locally finite) case in \([21]\).

It is convenient to consider also the Toeplitz extensions (cf. \([21]\))

\[
0 \longrightarrow I_S \longrightarrow T\mathcal{O}(E, S) \longrightarrow C^*(E) \longrightarrow 0,
\]
where $S$ is a subset of $E^0$ and $I_S = \bigoplus_{v \in S} K_v$, where $K_v$ is the algebra of compact operators on a Hilbert space of dimension $\#(P^* \cap r^{-1}(v))$. The $C^*$-algebra $T_0(E, S)$ is generated by operators $\{S_w \}_{w \in E^0}$ and $\{P_v \}_{v \in E^1}$ satisfying the conditions: $S_w^* S_w = P_{r(w)}$ and, for all $v \in S(E^1)$, $P_v \geq \sum_{w: s(w) = v} S_w^* S_w$, with equality for $v \in S$.

If $j : E \hookrightarrow \tilde{E}$ is an inclusion of directed graphs, the following functoriality property holds (cf. [21]): given $\tilde{S} \subset \tilde{E}$, consider the set $S$ of vertices $v$ in $E^0$ such that $j(v) \in \tilde{S}$ and the outgoing edges in $\tilde{E}^1$ with origin at $j(v)$ are all of the form $j(w)$, for some $w \in E^1$ with $s(w) = v$. Then this induces an injective $*$-homomorphism $J : T_0(E, S) \to T_0(\tilde{E}, \tilde{S})$.

In particular, for a family of subgraphs $E$ of a directed graph $\tilde{E}$, ordered by inclusions, with $\cup E^0 = \tilde{E}^0$ and $\cup E^1 = \tilde{E}^1$, and for a choice of $\tilde{S} \subset \tilde{E}^0$ and corresponding $S$ as above, we have

$$T_0(\tilde{E}, \tilde{S}) = \lim_{E} T_0(E, S). \quad (4.2)$$

4.1 Reduction graphs

In the theory of Mumford curves, it is important to consider also the reduction modulo powers $m^n$ of the maximal ideal $m \subset O_K$, which provides infinitesimal neighborhoods of order $n$ of the closed fiber. In the language of $C^*$-algebras, this corresponds to the following construction.

For each $n \geq 0$, we consider a subgraph $\Delta_{K,n}$ of the Bruhat-Tits tree $\Delta_K$ defined by setting

$$\Delta^0_{K,n} := \{v \in \Delta^0_K : d(v, \Delta^1) \leq n\}, \quad (4.3)$$

with respect to the distance (3.2), with $d(v, \Delta^1) : = \inf d(v, \tilde{v}) : \tilde{v} \in (\Delta^1)^0$, and

$$\Delta^1_{K,n} := \{w \in \Delta^1_K : s(w) \in \Delta^0_{K,n}\}. \quad (4.4)$$

Thus, we have $\Delta_{K,0} = \Delta^1$. For $n \geq 1$, the graph $\Delta_{K,n}$ has a nonempty set of sinks $\sigma_{K,n} \subset \partial \Delta_{K,n}$. We have $\Delta_K = \cup_n \Delta_{K,n}$.

For all $n \in \mathbb{N}$, the graph $\Delta_{K,n}$ is invariant under the action of the Schottky group $\Gamma$ on $\Delta$, and the finite graph $\Delta_{K,n}/\Gamma$ gives the dual graph of the reduction $X_K \otimes \mathbb{C}/m^{n+1}$. Thus, we refer to the $\Delta_{K,n}$ as reduction graphs. They form a directed family with inclusions $j_{n,m} : \Delta_{K,n} \hookrightarrow \Delta_{K,m}$, for all $m \geq n$, with all the inclusions compatible with the action of $\Gamma$.

For each reduction graph, we can consider corresponding $C^*$-algebras $C^*(\Delta_{K,n})$ and $C^*(\Delta_{K,n}/\Gamma) \simeq C^*(\Delta_{K,n}) \rtimes \Gamma$ (Morita equivalence).
The following result which is a direct application of the statements on graph C*-algebras listed in Section 4, describes the relation between the algebras C*(ΔK,n/Γ) and C*(ΔK/Γ).

**Lemma 4.1.** There are injective *-homomorphisms J_{n,m} : C*(ΔK,n) → C*(ΔK,m). Correspondingly, setting S_n = Δ_{K,n}^0 \setminus σ_{K,n}, yields

\[ C*(ΔK/Γ) = \lim_n TO(ΔK,n/Γ, S_n/Γ), \tag{4.5} \]

where TO(ΔK,n/Γ, S_n/Γ) satisfies

\[ 0 → \bigoplus_{v ∈ σ(ΔK,n/Γ)} X_v → TO(ΔK,n/Γ, S_n/Γ) → C*(ΔK,n/Γ) → 0. \tag{4.6} \]

## 5 Dynamics of walks on dual graphs

In this section, we introduce a dynamical system associated to the space \( W(Δ/Γ) \) of walks on the quotient of a directed tree \( Δ \) by a free action of \( Γ \). In particular, we are interested in the cases where \( Δ \) is (a certain extension of) one of the graphs \( Δ_{K,n} \) for some \( n ≥ 0 \). The same construction applies to the tree \( ΔΓ \), where this dynamical system is a subshift of finite type associated to the action of the Schottky group \( Γ \) on its limit set \( ΛΓ \), analogous to the one considered in [6].

Let \( V \subset Δ \) be a finite subtree whose set of edges consists of one representative for each \( Γ \)-class. This is a fundamental domain for \( Γ \) in the weak sense (following the notation of [12]), since some vertices may be identified under the action of \( Γ \). Correspondingly, we denote by \( V \) the set of ends of all infinite paths in \( Δ \) starting at points in \( \bar{V} \).

We assume that, for the \( Γ \)-invariant directed tree \( Δ \), the set \( V \) has finitely many edges. This is the case for \( ΔΓ \) as well as for any of the trees \( Δ_{K,n} \).

Since, for \( n ≥ 1 \), the graphs \( Δ_{K,n} \) have sinks, in order to consider the space of doubly infinite walks on these graphs, we need to complete each walk ending at a sink to an infinite walk obtained by repeating the last word. This is equivalent to extending the graph \( Δ_{K,n} \) by adding an infinite tail to each sink. Appending tails to sinks is a standard technique in the theory of graph C*-algebras, in order to reduce the general case to the easier case of graphs with no sinks. We use the notation \( Δ̃_{K,n} \) for the completed graph with infinite tails. These have an action of \( Γ \) obtained by extending the action on \( Δ_{K,n} \), by translating the whole tail in the same way as the corresponding sink in \( Δ_{K,n} \).

For \( Δ = Δ̃_{K,n} \), the set \( W(Δ/Γ) \) of doubly infinite walks on the graph \( Δ̃_{K,n}/Γ \) can be identified with the set of doubly infinite admissible sequences in the finite alphabet...
given by the edges in the fundamental domain $\tilde{V}$ of the graph $\Delta_{k,n}$, with both possible orientations.

On $\mathcal{W}(\Delta/\Gamma)$, we consider the topology generated by the sets $\mathcal{W}^n(\omega, \ell) = \{\tilde{w} \in \mathcal{W}(\Delta/\Gamma) : \tilde{w}_k = \omega_k, k \geq \ell\}$ and $\mathcal{W}^u(\omega, \ell) = \{\tilde{w} \in \mathcal{W}(\Delta/\Gamma) : \tilde{w}_k = \omega_k, k \leq \ell\}$, for $\omega \in \mathcal{W}(\Delta/\Gamma)$ and $\ell \in \mathbb{Z}$. With this topology, the space $\mathcal{W}(\Delta/\Gamma)$ is a totally disconnected compact Hausdorff space. The invertible shift map $T$, given by $(T\omega)_k = \omega_{k+1}$, is a homeomorphism of $\mathcal{W}(\Delta/\Gamma)$.

We have just described the dynamical system $(\mathcal{W}(\Delta/\Gamma), T)$ in terms of subshifts of finite type, according to the following definition.

**Definition 5.1.** A subshift of finite type $(S_A, T)$ consists of all doubly infinite sequences in the elements of a given finite set $W$ (alphabet) with the admissibility condition specified by a $\#W \times \#W$ elementary matrix,

$$S_A = \{(w_k)_{k \in \mathbb{Z}} : w_k \in W, A(w_k, w_{k+1}) = 1\}, \quad (5.1)$$

with the action of the invertible shift $(Tw)_k = w_{k+1}$.

**Lemma 5.2.** The space $\mathcal{W}(\Delta/\Gamma)$ with the action of the invertible shift $T$ is a subshift of finite type, where $\mathcal{W}(\Delta/\Gamma) = S_A$ with $A$ the directed edge matrix of the finite graph $\Delta/\Gamma$.

We can consider the mapping torus of $T$,

$$\mathcal{W}(\Delta/\Gamma)_T := \mathcal{W}(\Delta/\Gamma) \times [0, 1]/(Tx, 0) \sim (x, 1). \quad (5.2)$$

We consider the first cohomology group $H^1(\mathcal{W}(\Delta/\Gamma)_T, \mathbb{Z})$, identified with the group of homotopy classes of continuous maps of $\mathcal{W}(\Delta/\Gamma)_T$ to the circle. If we denote by $C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})$ the $\mathbb{Z}$-module of integer-valued continuous functions on $\mathcal{W}(\Delta/\Gamma)$, and by $C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T$ the cokernel of the map $\delta(f) = f - f \circ T$, we obtain the following result (cf. [2, 19]).

**Proposition 5.3.** The map $f \mapsto [\exp(2\pi \text{it} f(x))]$, which associates to an element $f \in C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})$ a homotopy class of maps from $\mathcal{W}(\Delta/\Gamma)_T$ to the circle, gives an isomorphism $C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T \cong H^1(\mathcal{W}(\Delta/\Gamma)_T, \mathbb{Z})$. Moreover, there is a filtration of $C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T$ by free $\mathbb{Z}$-modules $F_0 \subset F_1 \subset \cdots \subset F_n$, of rank $\theta_n - \theta_{n-1} + 1$, where $\theta_n$ is the number of admissible words of length $n + 1$ in the alphabet, so that
\[ H^1(\mathcal{W}(\Delta/\Gamma), \mathbb{Z}) = \lim_{\rightarrow} F_n. \]  

(5.3)

The quotients \( F_{n+1}/F_n \) are also torsion-free.

Proof. A continuous function \( f \in C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z}) \) depends on just finitely many coordinates \( \omega_k \) of \( \omega \in \mathcal{W}(\Delta/\Gamma) \). In particular, this implies that, for some \( k_0 \), the composite \( f \circ T^{k_0} \) is a function of only "future coordinates" (\( \omega_k \) with \( k \geq 0 \)). We denote by \( \mathcal{P} \subset C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z}) \) the submodule of functions of future coordinates. It is then clear that we have \( C(\mathcal{W}(\Delta/\Gamma), \mathbb{Z}) \) \( \mathcal{P} \). We also have an identification \( \mathcal{P} \simeq C(\mathcal{W}^+(\Delta/\Gamma), \mathbb{Z}) \). This gives a filtration \( \mathcal{P} = \sqcup_n P_n \), where \( P_n \) is identified with the submodule of \( C(\mathcal{W}^+(\Delta/\Gamma), \mathbb{Z}) \) generated by characteristic functions of \( \mathcal{W}^+(\Delta/\Gamma, p) \subset \mathcal{W}^+(\Delta/\Gamma) \), where \( p \in \mathcal{W}^+(\Delta/\Gamma) \) is a finite walk \( p = \omega_0 \cdots \omega_n \) of length \( n+1 \), and \( \mathcal{W}^+(\Delta/\Gamma, p) \) is the set of infinite paths \( \omega \in \mathcal{W}^+(\Delta/\Gamma) \), with \( \omega_k = \omega_k \) for \( 0 \leq k \leq n+1 \). We have \( \delta : P_n \to P_{n+1} \), with kernel the constant functions. We set \( F_n := P_n/\delta(P_{n-1}) \), for \( n \geq 1 \) and \( F_0 = P_0 \). The inclusions \( P_n \subset P_{n+1} \) induce injections \( j : F_n \to F_{n+1}, j(f + \delta P_{n-1}) = f + \delta P_n \), such that \( \delta(P) = \lim_{\rightarrow} F_n \). As \( \mathbb{Z} \)-modules, both the \( F_n \) and the quotients \( F_{n+1}/j(F_n) \) are torsion-free.

For more details, see the analogous argument given in [19, Theorem 19, pages 62–63].

5.1 The effect of field extensions

Let \( L \supset K \) be a finite extension, with branching index \( e_{L/K} \) and with \( f = [L : K]/e_{L/K} \). Then there is an embedding of the set of vertices \( \Delta_K^0 \subset \Delta_L^0 \). In between every two vertices of \( \Delta_K^0 \subset \Delta_L^0 \), there are \( e_{L/K} - 1 \) new vertices of \( \Delta_L \). If in \( \Delta_K \) every vertex has valence \( q + 1 \), then every vertex of \( \Delta_L \) has valence \( q^f + 1 \).

In particular, the image of the tree \( \Delta_L \subset \Delta_K \) in \( \Delta_L \) is the tree \( \Delta_L \subset \Delta_L \), whereas, when considering the tree \( \Delta_L^f \), we are inserting \( e_{L/K} - 1 \) new vertices in between each two vertices of \( \Delta_L^f \subset \Delta_K \). Notice that the algebras \( C^*(\Delta_L^f) \) and \( C^*(\Delta_L^f) \) are strongly Morita equivalent, since they both are equivalent to the commutative \( C^* \)-algebra \( C(\Lambda_L) \). Thus we have the following lemma.

**Lemma 5.4.** The strong Morita equivalence class of the graph \( C^* \)-algebras \( C^*(\Delta_L^f) \) and \( C^*(\Delta_L^f) \) is independent of finite-field extensions \( L \supset K \).  

The following example illustrates how the algebra \( O_{A^+} \) changes under field extensions, by showing the change in the edge matrix \( A^+ \). If the first directed graph shown in Figure 5.1 arises as dual graphs of the closed fiber for a totally split degenerate curve, then the effect on the graph of a field extension with \( e_{L/K} = 2 \) is illustrated in the second graph in Figure 5.1.
Figure 5.1 The effect of a field extension with $e_{L/K} = 2$.

The edge matrix of the original graph was of the form

$$A_+ = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad (5.4)$$

while the new edge matrix becomes

$$A_+ = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}. \quad (5.5)$$

We are interested in understanding the effect of field extensions on the construction considered in the previous paragraph. We have the following result.

**Proposition 5.5.** A finite-field extension $L \supset K$ determines a homomorphism

$$H^1\left(\mathcal{W}(\tilde{\Delta}_{L,n}/\Gamma)_T\right) \longrightarrow H^1\left(\mathcal{W}(\tilde{\Delta}_{K,n}/\Gamma)_T\right), \quad (5.6)$$

which is compatible with the filtrations. \qed

**Proof.** In [12], it is shown that there is a canonical choice of the fundamental domains $\tilde{V}$ and $V$ for the action of $\Gamma$ on the Bruhat-Tits tree $\Delta_K$ that is functorial under a finite extension $L \supset K$. This determines corresponding functorial choices of fundamental domains for the graphs $\Delta_{K,n}$. We obtain this way a corresponding embedding of walk spaces

$$J_{L,K,n} : \mathcal{W}(\tilde{\Delta}_{K,n}/\Gamma) \hookrightarrow \mathcal{W}(\tilde{\Delta}_{L,n}/\Gamma), \quad (5.7)$$
which replaces each edge in a sequence \( \omega \in W(\Delta_{K,n}/\Gamma) \) (an edge in \( \overline{V} \) for \( \Delta_{K,n} \)) with the corresponding \( e_{L/K} \) consecutive edges in the fundamental domain \( \overline{V} \) for \( \Delta_{L,n} \), thus obtaining an element in \( W(\Delta_{L,n}/\Gamma) \). This map satisfies \( J_{L,K,n} \circ T = T e_{L/K} \circ J_{L,K,n} \). Thus, we obtain an induced map

\[
J_{L,K,n,T} : W(\Delta_{K,n}/\Gamma)_T \rightarrow W(\Delta_{L,n}/\Gamma)_{T e_{L/K}},
\]

where, for \( \ell \geq 1 \), \( W(\Delta_{L,n}/\Gamma)_{T^\ell} \) denotes the mapping torus

\[
W(\Delta_{L,n}/\Gamma)_{T^\ell} \cong W(\Delta_{L,n}/\Gamma) \times [0, \ell]/(x, 0) \sim (T^\ell x, \ell),
\]

with a covering map \( \pi_\ell : W(\Delta_{L,n}/\Gamma)_{T^\ell} \rightarrow W(\Delta_{L,n}/\Gamma)_T \). Thus, we obtain a map

\[
\pi e_{L/K} \circ J_{L,K,n,T} : W(\Delta_{K,n}/\Gamma)_T \rightarrow W(\Delta_{L,n}/\Gamma)_{T e_{L/K}}.
\]

This induces a corresponding map in cohomology,

\[
(\pi e_{L/K} \circ J_{L,K,n,T})^* : H^1\left(W(\Delta_{L,n}/\Gamma)_T\right) \rightarrow H^1\left(W(\Delta_{K,n}/\Gamma)_T\right).
\]

To see this at the level of the filtrations of Proposition 5.3, notice that the map \( J_{L,K,n} \) also induces a restriction map

\[
r_{L,K,n} : C(W(\Delta_{L,n}/\Gamma), \mathbb{Z}) \rightarrow C(W(\Delta_{K,n}/\Gamma), \mathbb{Z}).
\]

If we denote by \( \mathcal{F}^{(L)}_N \) and \( \mathcal{F}^{(K)}_j \) the respective filtrations, then we have restriction maps \( r_{L,K,n} : \mathcal{F}^{(L)}_j \rightarrow \mathcal{F}^{(K)}_j \). If we denote by \( \delta_\ell \) the map \( \delta_\ell(f) = f - f \circ T^\ell \), then the restriction also satisfies \( r_{L,K,n} \circ \delta e_{L/K} = \delta \circ r_{L,K,n} \), hence there is an induced map \( r_{L,K,n} : F_j^{(L,e_{L/K})} \rightarrow F_j^{(K)} \), where we have set \( F_j^{(L,e_{L/K})} := \mathcal{F}^{(L)}_j / \delta e_{L/K} \mathcal{F}^{(L)}_{j-1} \). An argument analogous to the one used in the proof of Proposition 5.3 shows that the \( F_j^{(L,e_{L/K})} \) give a filtration of

\[
H^1\left(W(\Delta_{L,n}/\Gamma)_{T e_{L/K}}, \mathbb{Z}\right) = \lim_{\rightarrow j} F_j^{(L,e_{L/K})}.
\]

There is a corresponding map

\[
C\left(W(\Delta_{L,n}/\Gamma), \mathbb{Z}\right)_T \rightarrow C\left(W(\Delta_{L,n}/\Gamma), \mathbb{Z}\right)_{T e_{L/K}},
\]
induced by the covering \( \pi_{e/L/k} : \mathcal{W}(\Delta_{L,n}/\Gamma)_{T^{e/L/k}} \to \mathcal{W}(\Delta_{L,n}/\Gamma)_T \). On the level of filtrations, this has the following description. The module \( P_{j,e/L/k}^{(L)} \) can be identified with the span of functions in \( P_j^{(L)} \), \( s = 0, \ldots, e/L/k - 1 \), where \( P_j^{(L)} = T^j(P_j^{(L)}) \). The inclusion \( \iota : P_j^{(L)} \to P_{j,e/L/k}^{(L)} \) as \( P_{j,0}^{(L)} \) then satisfies \( \delta \circ \iota = \iota \circ \delta_{e/L/k} \). We obtain an induced map \( F_j^{(L)} \to F_j^{(L,e/L/k)} \). Thus, the map induced by the covering \( \pi_{e/L/k} \) also preserves the filtrations, and we obtain maps \( F_j^{(L)} \to F_j^{(K)} \) that induce \( (\pi_{e/L/k} \circ J_{L,K,T})^* \) on the direct limits.

5.2 Dynamical cohomology

Let \( \Delta \) be a directed tree on which the Schottky group \( \Gamma \) acts, with the same assumptions as in the previous paragraph.

On the set of ends \( \partial \Delta \), we consider a measure \( d\mu \) defined by first introducing the distance function \( d(v) := \text{dist}(v, x_0) \), for \( v \in \Delta^0 \) and \( x_0 \) the base point in \( \Delta^0 \) with respect to which the structure of directed graph on \( \Delta \) is determined. Then the measure on \( \partial \Delta \) is defined by assigning its value on the clopen set \( V(v) \), given by the ends of all paths in \( \Delta \) starting at a vertex \( v \), to be

\[
\mu(V(v)) = q^{-d(v)-1},
\]

with \( q = \text{card}(\mathcal{O}/m) \).

**Proposition 5.6.** This induces a measure on \( \mathcal{W}(\Delta/\Gamma) \), with respect to which the shift map \( T \) is measure preserving.

**Proof.** Notice that, if we identify the points of \( V(v) \) with infinite paths starting at \( v \),

\[
V(v) = \{w_0w_1 \cdots w_n \cdots : s(w_0) = v, w_k \in (\Delta)_+^1\},
\]

then the image \( T(V(v)) \) will have measure \( \mu(T(V(v))) = \mu(V(v))/q \). In the case of walks starting at a vertex \( v \), the map \( T \) scales the measure of the set of walks starting with an edge \( w \in (\Delta)_+^1 \) by a factor \( q^{-1} \) and the measure of the set of walks starting with an edge \( \bar{w} \) with \( \bar{w} \in (\Delta)_+^1 \) by a factor \( q \).

We can define a map from \( \mathcal{W}(\Delta/\Gamma) \) to \( V \times V \), by splitting each doubly infinite sequence

\[
\cdots w_{-n}w_{-n+1} \cdots w_{-1}w_0w_1 \cdots w_tw_{t+1} \cdots
\]

into the two sequences

\[
(w_0w_1 \cdots w_tw_{t+1} \cdots, w_{-1}w_{-2} \cdots w_{-n+1}w_{-n} \cdots),
\]

\[
(w_{-n}w_{-n+1} \cdots w_{-1}w_0w_1 \cdots)\]
each of which defines a point in the fundamental domain \( V \), if we identify admissible sequences of edges in the fundamental domain \( \Delta/\Gamma \) with admissible sequences of edges in \( \Delta \) with the condition that the first edge \( w_0 \) (or \( \bar{w}^{-1} \)) lies in a chosen fundamental domain \( \bar{V} \) of the action of \( \Gamma \) which contains the base point \( x_0 \). Then the action of \( T \) maps (5.18) to

\[
(w_1 \cdots w_{\ell}w_{\ell+1} \cdots, \bar{w}_0\bar{w}_{-1}\bar{w}_{-2} \cdots \bar{w}_{-n+1}\bar{w}_{-n} \cdots),
\]

hence it scales the measure on one factor by \( q \) and on the other factor by \( q^{-1} \), so that the measure induced on \( \mathcal{W}(\Delta/\Gamma) \), by restricting to \( V \times V \) the product measure on \( \partial \Delta \times \partial \Delta \), is preserved by \( T \).

Consider the free \( \mathbb{Z} \)-modules \( \mathcal{P}_n \), introduced in the proof of Proposition 5.3, of functions of at most \( n+1 \) future coordinates. We can realize \( \mathcal{P}_n \otimes \mathbb{C} \) as a vector subspace of \( L^2(\partial \Delta, d\mu) \). The operator \( \delta \) is bounded in norm, hence the \( F_n = \mathcal{P}_n / \delta \mathcal{P}_{n-1} \) have induced norms and bounded inclusions \( F_n \otimes \mathbb{C} \hookrightarrow F_{n+1} \otimes \mathbb{C} \), where the \( F_n \) are the torsion-free \( \mathbb{Z} \)-modules of Proposition 5.3.

Using the inner product induced from \( L^2(\partial \Delta, d\mu) \), we can split

\[
F_n \cong F_{n-1} \oplus (F_{n-1}^\perp \cap F_n).
\]

Definition 5.7. The dynamical cohomology \( H^1_{\text{dyn}}(\Delta/\Gamma) \) is the norm completion of the graded complex vector space

\[
H^1_{\text{dyn}}(\Delta/\Gamma) := \oplus_{k \geq 0} \text{Gr}_k
\]

with \( \text{Gr}_n := (F_n \cap F_{n-1}^\perp) \).

We now construct a representation, by linear bounded operators, of a graph \( C^* \)-algebra on this Hilbert space.

Proposition 5.8. There is a representation of the algebra \( C^*(\Delta/\Gamma) \) by bounded linear operators on the Hilbert space \( H^1_{\text{dyn}}(\Delta/\Gamma) \).

Proof. For \( w \in (\Delta/\Gamma)_+ \), we define linear operators \( T_w \) of the form

\[
(T_w f)(w_0w_1w_2 \cdots) = \begin{cases} f(ww_0w_1w_2 \cdots), & r(w) = s(w_0), \\ 0, & r(w) \neq s(w_0). \end{cases}
\]
For $v \in \Delta/\Gamma^0$, let $P_v$ denote the projection

$$
(P_v f)(w_0w_1w_2 \cdots) = \begin{cases} f(w_0w_1w_2 \cdots), & s(w_0) = v, \\ 0, & s(w_0) \neq v. \end{cases}
$$

We also define projections $\Pi_w$ of the form

$$
(\Pi_w f)(w_0w_1w_2 \cdots) = \begin{cases} f(w_0w_1w_2 \cdots), & w_0 = w, \\ 0, & w_0 \neq w. \end{cases}
$$

Finally, we define linear operators $s_w$ and $S_w$

$$s_w := \sum_{w'} A(w,w')T_w\Pi_{w'}, \quad S_w := \sqrt{q} s_w.
$$

These define bounded linear operators on $L^2(\partial \Delta, d\mu)$. The $S_w$ are partial isometries, satisfying $S_w S_w^* = \Pi_w$. Thus, the operators $\{P_v, S_w\}$ form a Cuntz-Krieger family, since we obtain

$$
P_v = \sum_{s(w)=v} \Pi_w = \sum_{s(w)=v} S_w S_w^*,
$$

$$S_w^* S_w = \sum A(w,w')S_w S_w^* = \sum A(w,w')P_{w'} = P_{r(w)}.
$$

The same argument given in [6, Proposition 4.19], shows that $S_w \delta = \delta S_w$, so that they descend to bounded operators on $\mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma)$. ■

5.3 Embedding of cohomologies

Let $\Delta$ be any of the trees $\check{\Delta}_k, n \geq 0$, or $\Delta_0$. We construct a family of embeddings of the cohomology of the dual graph $\Delta_0/\Gamma$ in the dynamical cohomology $\mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma)$. Let $|\Delta|$ denote the geometric realization of the dual graph $\Delta_0/\Gamma$.

**Theorem 5.9.** For each $N \geq 0$, there are embeddings $\phi_N$ of the cohomology of the dual graph $H^1(\Delta_0/\Gamma, \mathbb{C})$ into the dynamical cohomology $\mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma)$. □

**Proof.** We define maps $\phi_N$ from the cohomology $H^1(\Delta_0/\Gamma, \mathbb{C})$ to $\mathcal{H}^1_{\text{dyn}}(\Delta_k/\Gamma)$ in the following way: let $\{\gamma_i\}_{i=1}^N$ be a chosen set of generators of the Schottky group $\Gamma$, $[\gamma_i]$ the corresponding homology classes in $H_1(\Delta_0/\Gamma, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$, and $\eta_i$ the dual generators in cohomology. Let $w_i$ be the finite admissible word in the edges of the directed graph $\bar{V}$ (the fundamental domain for the $\Gamma$-action on $\Delta$) that represents the generator $\gamma_i$, with length
\[ \ell_i = |w_i|. \] Here, in the case of \( \Delta = \tilde{\Delta}_{K,n} \), we first notice that the first cohomology group of the dual graph is the same as \( H^1(|\Delta^\ell_f/\Gamma|, \mathbb{Z}) \), since the insertion of extra vertices does not change the topology of the graph. Since we have \( \Delta^\ell_f \subset \Delta_{K,n} \), we obtain the \( w_i \) as above, as edges in the corresponding fundamental domain \( \tilde{V} \) for \( \Delta_{K,n} \).

We then set
\[
\Phi_N(\eta_i) = P_{\ell_i} \chi_{i,N}, \quad (5.27)
\]
where
\[
\chi_{i,N} := \chi_{W^+(\Delta/\Gamma, w_i \cdots w_i)}^{N \text{ times}} \quad (5.28)
\]
is the characteristic function of the set \( W^+(\Delta/\Gamma, w_i \cdots w_i) \) of walks in \( \Delta/\Gamma \) that begin with the word \( w_i \) repeated \( N \)-times. The elements \( \chi_{i,N} \) lie in \( F_{\ell_i} \). We denote by \( P_k^\perp \) the orthogonal projection of \( F_k \) onto \( G_k \).

The elements \( \phi_N(\eta_i) \) are all linearly independent in \( H^1_{\text{dyn}}(\Delta/\Gamma) \), hence the \( \phi_N \) give linear embeddings of \( H^1(|\Delta^\ell_f/\Gamma|, \mathbb{C}) \) into \( H^1_{\text{dyn}}(\Delta/\Gamma) \).

**Corollary 5.10.** For any finite set of distinct \( \{N_k\} \), the set \( \bigcup_k \{\phi_{N_k}(\eta_i)\}_{i=1}^a \) consists of linearly independent vectors in \( H^1_{\text{dyn}}(\Delta/\Gamma) \). Thus, this gives an embedding
\[
\Phi = \bigoplus_N \phi_N : \bigoplus_N H^1(|\Delta^\ell_f/\Gamma|, \mathbb{C}) \longrightarrow H^1_{\text{dyn}}(\Delta/\Gamma) . \quad (5.29)
\]

Notice that the map \( \Phi \) of (5.29) does not preserve the graded pieces, due to the rescaling of the degrees by \( \ell_i \) in (5.27). This has to be taken into account if we want to recover arithmetic information such as the local L-factors of [8] from the dynamical cohomology. For this reason, it may be necessary to blow up some double points on the special fiber.

**Lemma 5.11.** It is always possible to reduce to the case where all the \( \ell_i = \ell \), after blowing up a certain number of double points on the special fiber.

**Proof.** Blowing up a double point on the special fiber \( C(k) \), for \( k = \mathcal{O}/m \), corresponds to introducing one extra vertex in the dual graph \( \Delta^\ell_f/\Gamma \). This changes, by one, the lengths \( \ell_i \) for those generators of \( \Gamma \) for which the corresponding chain of edges \( w_i \) passes through the newly inserted vertex. \( \blacksquare \)
Thus, possibly after blowing up some double points, we obtain

$$
\phi_N : H^1 \left( |\Delta r/\Gamma|, \mathbb{C} \right) \hookrightarrow Gr N \ell \subset H^1_{\text{dyn}}(\Delta K/\Gamma),
$$

for some $\ell \geq 1$.

### 5.4 Spectral triples

In this paragraph, we show that we can associate to a given Mumford curve a family of spectral triples $(A_n, \mathcal{H}_n, D_n)$, for $n \geq -1$. Each triple in this family corresponds to the choice of a graph $\Delta = \Delta_{K,n}$, for $n \geq 0$ and $\Delta = \Delta_r$ for $n = -1$. The Dirac operator in these spectral triples depends only on the graded structure of the space $H^1_{\text{dyn}}(\Delta/\Gamma)$.

Recall that a spectral triple $(A, \mathcal{H}, D)$ is a triple of a C*-algebra, with a representation in the algebra of bounded operators on the Hilbert space $\mathcal{H}$, and a Dirac operator $D$, which is a selfadjoint unbounded operator acting on $\mathcal{H}$, such that $(\lambda - D)^{-1}$ is a compact operator for all $\lambda \notin \mathbb{R}$ and the commutators $[a, D]$ are bounded operators for all $a \in A_0$, with $A_0$ a dense involutive subalgebra of $A$.

Recall also that a spectral triple $(A, \mathcal{H}, D)$ has an associated family of zeta functions of the form

$$
\zeta_{\alpha,|D|}(z) = \text{Tr} \left( \alpha|D|^{-z} \right) = \sum_{\lambda \in \text{Spec}(|D|) \backslash \{0\}} \text{Tr} \left( \alpha \Pi_\lambda \right) \lambda^{-z},
$$

with $\alpha \in A_0 \cup [D, A_0]$, and with $\Pi_\lambda$ the spectral projection on $\lambda \in \text{Spec}(|D|)$. The properties of these zeta functions are related to the notion of dimension spectrum for the spectral triple (the set of poles of the $\zeta_{\alpha,|D|}$) and to the local index formula of Connes and Moscovici.

There are corresponding two-variable zeta functions

$$
\zeta_{\alpha,|D|}(s, z) = \sum_{\lambda \in \text{Spec}(|D|)} \text{Tr} \left( \alpha \Pi_\lambda \right) (s + \lambda)^{-z},
$$

and associated regularized determinants

$$
\det_{\infty, |A, D|} (s) = \exp \left( - \frac{d}{dz} \zeta_{\alpha,|D|}(s, z) |_{z=0} \right).
$$

We consider the Hilbert space

$$
\mathcal{H} = \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma) \oplus \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma).
$$
On this space, we consider the diagonal action of \( C^*(\Delta/\Gamma) \). We also introduce the notation \( \text{Gr}_{n,-} := \text{Gr}_n \oplus 0 \) and \( \text{Gr}_{n,+} := 0 \oplus \text{Gr}_n \).

We define the Dirac operator \( D \) acting on \( \mathcal{H} \) by setting

\[
D|_{\text{Gr}_{+,n}} = n, \quad D|_{\text{Gr}_{-,n}} = -(n + 1).
\]

**Proposition 5.12.** The data \((C^*(\Delta/\Gamma), \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma) \oplus \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma), D)\) determines a spectral triple in the sense of Connes [4].

**Proof.** In order to obtain a spectral triple, we need to check the compatibility requirement between the Dirac operator \( D \) and the action of the algebra \( C^*(\Delta/\Gamma) \). It is sufficient to check that the commutators \([D, S_w]\) are bounded operators. This follows easily since \( S_w : \text{Gr}_{\pm,n} \to \text{Gr}_{\pm,n-1} \), so that \([D, S_w]f = \mp S_w f\). The remaining properties are easily verified. \( \blacksquare \)

We can modify slightly the above construction, in order to take into account the scaling factor \( \ell \) in the grading between \( \oplus \mathcal{N} H^1(\Delta/\Gamma, \mathbb{C}) \) and its image in \( \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma) \) under \( \Phi \).

**Corollary 5.13.** The operator \( D \) of (5.35) can be modified by setting

\[
D|_{\text{Gr}_{+,n}} = \frac{n}{\ell} \frac{2\pi}{\log q}, \quad D|_{\text{Gr}_{-,n}} = -(n + 1) \frac{2\pi}{\ell \log q}.
\]

Here, \( q \) is the cardinality of the residue field \( k(\nu) \) and \( \ell \) is the length of all the words representing the generators of \( \Gamma \), possibly after blowing up some points on the special fiber. This modified operator \( D \) still gives a spectral triple

\[
(C^*(\Delta/\Gamma), \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma) \oplus \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma), D).
\]

**5.5 Local \( L \)-factor**

Let \( X \) be a curve over a global field \( K \). We assume semistability at all places of bad reduction. The local Euler factor at a place \( \nu \) has the following description (see [20]):

\[
L_{\nu}(H^1(X), s) = \det \left( 1 - Fr^*_{\nu} N(\nu)^{-s} |H^1(X, \mathbb{Q}_\ell)^{I_{\nu}}|^{-1} \right)^{-1}.
\]

Here, \( Fr^*_{\nu} \) is the geometric Frobenius acting on \( \ell \)-adic cohomology of \( \bar{X} = X \otimes \text{Spec}(\bar{K}) \), with \( \bar{K} \) an algebraic closure and \( \ell \) a prime with \((\ell, q) = 1\), where \( q \) is the cardinality of the
residue field $k(v)$ at $v$. We denote by $N$ the norm map. The determinant is evaluated on the inertia invariants $H^1(X, \mathbb{Q}_\ell)^v$ at $v$ (all of $H^1(X, \mathbb{Q}_\ell)$ when $v$ is a place of good reduction).

Suppose $v$ is a place of $k(v)$-split degenerate reduction. Then the completion of $X$ at $v$ is a Mumford curve $X_\Gamma$. In this case, the Euler factor (5.38) takes the following form:

$$L_v(H^1(X_\Gamma), s) = \prod_{\lambda} (1 - \lambda q^{-s})^{-\dim H^1(X_\Gamma)^v_{\lambda}} = (1 - q^{-s})^{-g},$$

(5.39)

since the eigenvalues $\{\lambda\}$ of the Frobenius, in this case, are all $\lambda = 1$. Here, we denote by $H^1(X_\Gamma)^v_{\lambda}$ the eigenspaces of the Frobenius.

Deninger in [8, 9] obtained the local factor (5.38) as a regularized determinant over an infinite-dimensional cohomological theory.

In the case of Mumford curves, Deninger’s calculation can be recast in terms of the data of the spectral triples of Proposition 5.12 and of the embeddings $\Phi^\pm$ of cohomologies.

We consider the operator $iD$ (an imaginary rotation of the Dirac operator) and the zeta functions

$$\zeta_{a,iD,+}(s, z) := \sum_{\lambda \in \text{Spec}(iD) \cap \{0, \infty\}} \text{Tr} \left( a\Pi_{\lambda} \right) (s + \lambda)^{-z},$$

$$\zeta_{a,iD,-}(s, z) := \sum_{\lambda \in \text{Spec}(iD) \cap \{-\infty, 0\}} \text{Tr} \left( a\Pi_{\lambda} \right) (s + \lambda)^{-z}. \quad (5.40)$$

Then we have a regularized determinant

$$\det_{\infty, a,iD} (s) := \exp \left( -\zeta'_{a,iD,+}(s, 0) \right) \exp \left( -\zeta'_{a,iD,-}(s, 0) \right). \quad (5.41)$$

**Theorem 5.14.** Let $\pi(V)$ be the orthogonal projection of $\mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma) \oplus \mathcal{H}^1_{\text{dyn}}(\Delta/\Gamma)$ onto the graded subspace $V = \text{Im}(\Phi^-) \oplus \text{Im}(\Phi^+)$, where $\Phi^\pm$ denote the maps $\Phi \oplus \Phi$ and $\Phi \oplus 0$. Then the regularized determinant (5.41), with $a = \pi(V)$ and $D$ the Dirac operator of Corollary 5.13 satisfies

$$\det_{\infty, \pi(V), iD} (s) = L_v(H^1(X_\Gamma), s)^{-1}. \quad (5.42)$$
Proof. When we compute the zeta functions (5.40) for the Dirac operator of Corollary 5.13, and \( a = \pi(\mathcal{V}) \), we obtain

\[
\zeta_{\pi(\mathcal{V}), iD,+}(s, z) = \sum_{n=0}^{\infty} \text{Tr} \left( \pi(\mathcal{V}) \Pi_{+,nt} \right) (\gamma(\tau + n))^z,
\]

\[
\zeta_{\pi(\mathcal{V}), iD,-}(s, z) = \sum_{n=0}^{\infty} \text{Tr} \left( \pi(\mathcal{V}) \Pi_{-,nt} \right) (\gamma(\tau - n))^z - \text{Tr} \left( \pi(\mathcal{V}) \Pi_{-,0} \right) (\gamma)^z,
\]

for \( \gamma = 2\pi i / \log q \) and \( \tau = (\log q / (2\pi i)) s \), with choice of arguments \(-\pi < \arg \gamma(\tau + n) < \pi\), as in [9].

Furthermore, we have \( \text{Tr}(\pi(\mathcal{V}) \Pi_{\pm,nt}) = \dim(\text{Gr}_{\pm,nt} \cap \mathcal{V}) = g \). In fact, the space \( \text{Gr}_{+,nt} \cap \mathcal{V} = \text{Im}(\Phi^+) \) is generated by \( 0 \oplus \chi_{i,n} \) for \( i = 1, \ldots, g \) and \( \text{Gr}_{-,nt} \cap \mathcal{V} = \text{Im}(\Phi^-) \) is generated by the element \( \chi_{i,n} \oplus 0 \) for \( i = 1, \ldots, g \).

The result then follows the calculation of the regularized determinant given in [9]. Namely, we obtain

\[
(1 - q^{-s})^{-g} = (\tau \gamma)^{-g} \exp \left( -g \zeta'_{\gamma}(\tau, 0) \right) \exp \left( -g \zeta'_{-\gamma}(-\tau, 0) \right),
\]

which is exactly the regularized determinant \( \det_{\infty}(s - \Theta_q) \) computed in [9] for the spectrum (with multiplicity \( g \))

\[
\text{Spec}(s - \Theta_q) = \left\{ \frac{2\pi i}{\log q} \left( \frac{s \log q}{2\pi i} + n \right) : n \in \mathbb{Z} \right\}.
\]

It is very interesting to notice an important difference between the Archimedean and non-Archimedean cases. At the Archimedean prime (cf. [6, 7]), the local factor is described in terms of zeta functions for a Dirac operator \( D \). On the other hand, at the non-Archimedean places, in order to get the correct normalization as in [9], we need to introduce a rotation of the Dirac operator by the imaginary unit, \( D \rightarrow iD \). This rotation corresponds to the Wick rotation that moves poles on the real line to poles on the imaginary line (zeroes for the local factor) and appears to be a manifestation of a rotation from Minkowskian to Euclidean signature \( \text{it} \rightarrow \text{t} \), as already remarked by Manin ([14, page 135]), who wrote that imaginary time motion may be held responsible for the fact that zeroes of \( \Gamma(s)^{-1} \) are purely real whereas the zeroes of all non-Archimedean Euler factors are purely imaginary. This seems to hint to the existence of a more refined construction involving Minkowskian geometry, where the rotation \( D \rightarrow iD \) could be interpreted as a rotation \( \text{it} \rightarrow \text{t} \) of an infinitesimal length element \( D^{-1} \sim ic \text{dt} \). A more precise treatment...
would require adapting the structure of spectral triple to the case of Minkowskian signature. Another piece of supporting evidence for the idea that a more refined construction should involve Minkowskian geometry comes from the cohomological construction of [5]. In fact, in [6], we only used part of the full symmetry group determined by the Lefschetz module structure, namely the part corresponding to real hyperbolic geometry, so as to match the results of [13]. The full symmetry group is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, which is the isometry group not of 3-dimensional real hyperbolic geometry, but of its Minkowskian version, anti de Sitter space $AdS_2+1$. On the relation between the results of [13] and $AdS_2+1$, see also [16].

5.6 AF core

In order to understand the zeta functions $\zeta_{\pi(\mathcal{V}),D,\pm}(\tau, z)$ of Theorem 5.14 in terms of the spectral triple, we still need to express the operator $\pi(\mathcal{V})$ in terms of elements in the algebra $C^*(\Delta/\Gamma)$. The graph $C^*$-algebra $C^*(\Delta/\Gamma)$ contains an AF core given by the AF algebra obtained as the norm closure of $\bigcup_n \mathcal{F}_n$, where the finite-dimensional algebras $\mathcal{F}_n$ are given by

$$\mathcal{F}_n = \text{span} \{ S_\mu S_\nu^* : \mu, \nu \in \mathcal{P}^n(\Delta/\Gamma), \ t(\mu) = t(\nu) \}. \quad (5.46)$$

Here, we used the notation $S_\mu = S_{w_1} \cdots S_{w_k}$, for $\mu = w_1 \cdots w_k$. The AF core can be identified with the fixed-point algebra of the gauge action (cf. [1]),

$$\bigcup_n \mathcal{F}_n \cong C^*(\Delta/\Gamma)^{U(1)}. \quad (5.47)$$

Let $[w_i]_{i=1}^p$ be the words corresponding to the generators of $\Gamma$, which we assume all of equal length $\ell$, possibly after some blowups. Let $S_{w_i}$ be the corresponding operators in $C^*(\Delta/\Gamma)$. The operators $S^n_{w_i} S^*_n$ belong to the subalgebra $\mathcal{F}_{n,\ell}$ in the AF core of $C^*(\Delta/\Gamma)$.

Each $Q_{i,n} = S^n_{w_i} S^*_n$ acts on $L^2(\partial\Delta, d\mu)$ as multiplication by the characteristic function $\chi_{i,n}$ of (5.28), hence $Q_{i,n}$ maps $Gr_{+,n,\ell}$ to itself, with range the one-dimensional subspace of $Gr_{+,n,\ell}$ spanned by $\chi_{i,n}$. Thus, the operator $Q_n = \sum_i Q_{i,n}$ projects $Gr_{+,n,\ell}$ onto the $g$-dimensional subspace $Gr_{+,n,\ell} \cap \mathcal{V}$.

For the Dirac operator of (5.36), we write $D = \sum_{n \geq 0} \Pi_n \lambda_n$, where with $\lambda_n = 2\pi n/\ell \log q$ and $\Pi_n = \Pi_{+,n} \oplus \Pi_{-,n-1}$. We then have the following result.
Proposition 5.15. The zeta function

\[ \zeta_{\alpha(V),|D|}(z) = \text{Tr} \left( \pi(V)|D|^{-z} \right) \]  

(5.48)

can be written in the form

\[ \zeta_{\alpha(V),|D|}(z) = \text{Tr} \left( \sum_{n>0} Q_n \Pi_n \lambda^{-z} \right) \]  

(5.49)

with the \( Q_n \) in the AF core of \( \mathbb{C}^*(\Delta/\Gamma) \).

6 Foam spaces

We now consider the local factor (5.38) in the more general case, where we drop the assumption that \( v \) is a place of \( k(v) \)-split degenerate reduction. In this case, we no longer have a \( p \)-adic uniformization of the completion of \( X \) at \( v \) as a Mumford curve, and correspondingly, the local factor is no longer determined solely in terms of the combinatorics of the dual graph, but it depends essentially on extra geometric information on the nature of the degeneration. In particular, the inertia invariants \( H^1(\bar{X}, \mathbb{Q}_\ell)^I_v \) are described only partly by the cohomology of the dual graph, with the extra information provided by the cohomology of the single components of the dual fiber, which in the general case will no longer be just rational curves.

More precisely, if we denote by \( H^1(\bar{X})^I_\lambda \) the eigenspace of the geometric Frobenius, with eigenvalue \( \lambda \), we can write the Euler factor in the form

\[ L_v(H^1(X),s) = \prod_\lambda (1 - \lambda q^{-s})^{-d_\lambda}, \]  

(6.1)

with \( d_\lambda = \dim H^1(\bar{X})^I_\lambda \). Deninger’s description of the local factor as regularized determinant holds in this more general case, in the form \( \det_{\infty} (s-\Theta) \) where \( \Theta \) has spectrum \( \{ \alpha_\lambda + 2\pi i n / \log q \} \), with \( n \in \mathbb{Z}, \lambda \in \text{Spec}(\text{Fr}_v^*) \), and \( q^{\alpha_\lambda} = \lambda \).

We want to modify the graphs \( \Delta/\Gamma \) considered in the previous sections, in such a way that the corresponding dynamical cohomology will contain a linear subspace isomorphic to an infinite direct sum \( \oplus \mathbb{N} H^1(\bar{X}, \mathbb{Q}_\ell)^I_\lambda \), and such that the construction of the spectral triple and the derivation of the regularized determinant described in the case of Mumford curves will extend to this case to recover (6.1).

Using the exact sequence of [17, pages 110–111], we obtain an identification

\[ H^1(\bar{X}, \mathbb{Q}_\ell)^I_v \otimes_{\sigma_v} \mathbb{C} = H^1(\Delta/\Gamma) \oplus H^1(X_v(0)), \]  

(6.2)
where \( \sigma_\ell : Q_\ell \to \mathbb{C} \) is a fixed embedding of \( Q_\ell \) in \( \mathbb{C} \), for a prime \( \ell \) with \((\ell, q) = 1\), and we denote by \( X^{(0)}_\ell \) the disjoint union of the components of the special fiber. In the case of \( k(p) \)-split reduction, where all components are \( \mathbb{P}^1 \)'s, (6.2) is simply identified with \( H^1((\Delta_\ell / \Gamma)) \) as in the previous sections. The finite decomposition \( H^1(X, Q_\ell)^{I_\nu} = \oplus_\lambda H^1(X, Q_\ell)^{I_\nu}_\lambda \) in eigenvalues of the geometric Frobenius provides corresponding spaces \( H^1((\Delta_\ell / \Gamma))_\lambda \) and \( H^1(X^{(0)}_\ell)_\lambda \) of dimensions \( d^1_\lambda \) and \( d^0_\lambda \), respectively, with \( d^1_\lambda + d^0_\lambda = d_\lambda \).

We choose vertices \( \chi_\lambda \) (not necessarily distinct) of \( \Delta_\ell / \Gamma \), and attach to the vertex \( \chi_\lambda \) new outgoing edges \( \omega_{i,\lambda} \), with \( i = 1, \ldots, d^0_\lambda \). We denote by \( E_\nu \) the oriented graph obtained via this construction, after appending tails to all sinks.

Remark 6.1. For certain classes of examples, our graph \( E_\nu \) can be embedded as a subgraph of the “foam space” defined in [3]. The foam space is a graph \( F_\nu \) associated to the fiber \( X_\nu \) of an arithmetic surface \( X \) over Spec\((\mathcal{O}_K)\), obtained by replacing the special fiber \( X_\nu \) with an infinite series of blowups of its \( F_q \)-points. The graph \( F_\nu \) is the limit of the dual graphs associated to this series of blowups (cf. [15, Section 36]). For this reason, we think of the graphs \( E_\nu \) as a generalization of “foam spaces.”

To our foam space \( E_\nu \), we associate the corresponding dynamical cohomology \( \mathcal{H}^1(E_\nu) \) as in the previous sections, and the graph \( C^* \)-algebra \( C^*(E_\nu) \). The argument of Proposition 5.12 extends to this case and gives a spectral triple

\[
(C^*(E_\nu), \mathcal{H}^1(E_\nu) \oplus \mathcal{H}^1(E_\nu), D). \tag{6.3}
\]

We now define embeddings of cohomology groups as follows. Let \( \omega_{i,\lambda} \), for \( i = 1, \ldots, d^1_\lambda \) be loops of edges in \( \Delta_\ell / \Gamma \), with \( |\omega_{i,\lambda}| = \ell_{i,\lambda} \), representing homology classes dual to a basis \( \{\eta^r_{i,\lambda}\} \) of \( H^1((\Delta_\ell / \Gamma))_\lambda \). Up to adding vertices to the graph \( \Delta_\ell / \Gamma \) by blowing up double points in the closed fiber, we can assume that all the \( \ell_{i,\lambda} = \ell \). Adding vertices in this way does not change \( H^1(X^{(0)}_\ell) \), since the components of the closed fiber that correspond to the new vertices all have trivial \( H^1 \).

We consider then the linear embedding

\[
\Phi^r_{N,\lambda} : H^1((\Delta_\ell / \Gamma))_\lambda \to \text{Gr}_{N\ell} \subset \mathcal{H}^1(E_\nu), \tag{6.4}
\]

given by

\[
\Phi^r_{N,\lambda} (\eta^r_{i,\lambda}) = p_{\ell_{i,\lambda}} \cdot \mathcal{X}^{(0)}_\nu (\omega_{i,\lambda} \cdots \omega_{i,\lambda})_N \text{.} \tag{6.5}
\]
We also consider the embeddings
\[
\Phi_{N,\lambda}^0 : \mathcal{H}_0^1(X_\lambda^0, \lambda) \hookrightarrow \text{Gr}_{Nt} \subset \mathcal{H}_0^1(E_v),
\]
where the \( \eta_{N,\lambda}^0 \) form a basis of \( \mathcal{H}_0^1(X_\lambda^0, \lambda) \) and the \( w_{N,\lambda}^0 \) are the corresponding oriented edges of \( E_v \). Let \( \Phi_{N,\lambda}^+ = \oplus_N \Phi_{N,\lambda}^0 \) and \( \Phi_{N,\lambda}^- = \oplus_N \Phi_{N,\lambda}^0 \), and let \( \Phi_{\lambda} = \Phi_{\lambda}^+ \oplus \Phi_{\lambda}^- \). With \( \Phi_{\lambda}^\pm \) defined as the \( \Phi_{\lambda} \) in \text{Theorem 5.14}, we denote by \( \mathcal{V}_{\lambda} = \text{Im}(\Phi_{\lambda}^-) \oplus \text{Im}(\Phi_{\lambda}^+) \), and by \( \pi(\mathcal{V}_{\lambda}) \) the corresponding orthogonal projection.

We then extend the result of \text{Theorem 5.14} to this more general setting.

\textbf{Theorem 6.2.} Consider the regularized determinants (5.41), with \( a_{\lambda} = \pi(\mathcal{V}_{\lambda}) \) and \( D \) the Dirac operator of \text{Corollary 5.13} for the spectral triple \( (C^*(E_v), \mathcal{H}_0^1(E_v) \oplus \mathcal{H}_0^1(E_v), D) \). This gives

\[
\prod_{\lambda} \det_{\infty, \pi(\mathcal{V}_{\lambda}), \text{id}} (s) = L_v(H_0^1(X), s)^{-1}. \tag{6.7}
\]

The operators \( \pi(\mathcal{V}_{\lambda}) \) are related to elements in the AF core of the \( C^* \)-algebra \( C^*(E_v) \) as in (5.49).

\textbf{Proof.} We compute the zeta functions (5.40) for the Dirac operator of the spectral triple \( (C^*(E_v), \mathcal{H}_0^1(E_v) \oplus \mathcal{H}_0^1(E_v), D) \), with \( a = \pi(\mathcal{V}) \). We obtain

\[
\zeta_{\pi(\mathcal{V}_{\lambda}), \text{id}, +}(s, z) = \sum_{n=0}^{\infty} \text{Tr} (\pi(\mathcal{V}_{\lambda}) \Pi_{+, n\ell}) (\gamma (\tau_{\lambda} + n))^{-z},
\]

\[
\zeta_{\pi(\mathcal{V}_{\lambda}), \text{id}, -}(s, z) = \sum_{n=0}^{\infty} \text{Tr} (\pi(\mathcal{V}_{\lambda}) \Pi_{-, n\ell}) (\gamma (\tau_{\lambda} - n))^{-z} - \text{Tr} (\pi(\mathcal{V}_{\lambda}) \Pi_{-, 0}) (\tau_{\lambda} \gamma)^{-z}, \tag{6.8}
\]

for \( \gamma = 2\pi i / \log q \), \( \tau_{\lambda} = (\log q / (2\pi i))(s - a_{\lambda}) \), and \( q^{a_{\lambda}} = \lambda \), and with choice of arguments as in [9].

By construction, we have \( \text{Tr}(\pi(\mathcal{V}_{\lambda}) \Pi_{\pm, n\ell}) = \dim(\text{Gr}_{\pm, n\ell} \cap \mathcal{V}_{\lambda}) = d_{\lambda} \), hence the left-hand side of (6.7) is the regularized determinant \( \det_{\infty, \Theta_q} (s - \Theta_q) \) computed in [9], with spectrum (with multiplicities \( d_{\lambda} \))

\[
\text{Spec} (s - \Theta_q) = \left\{ \frac{2\pi i}{\log q} \left( \log q / (2\pi i) + n \right) : n \in \mathbb{Z}, \lambda \in \text{Spec} \left( \text{Fr}_v \right) \right\}. \tag{6.9}
\]

The expression of the operators \( \pi(\mathcal{V}_{\lambda}) \) in terms of operators in the AF core of the \( C^* \)-algebra \( C^*(E_v) \) is analogous to the case of Mumford curves. \qed
Acknowledgments

Part of this work was done during visits of the first author to the Max Planck Institute in Bonn and of the second author to Florida State University and University of Toronto. We thank these institutions for hospitality and support. We are very grateful to Christopher Deninger for a crucial remark about normalizations. This research has been partially supported by NSERC Grant 72016789 and by Humboldt Foundation and the German Government (Sofja Kovalevskaya Award).

References

1972 C. Consani and M. Marcolli


Caterina Consani: Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3
E-mail address: kc@math.toronto.edu

Matilde Marcolli: Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: marcolli@mpim-bonn.mpg.de