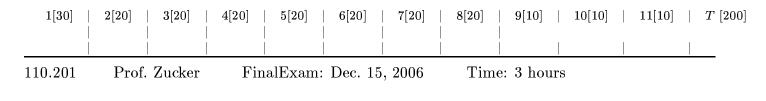
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ETHICS PLEDGE: I agree to complete this exam without unauthorized assistance from any

person or person's work, materials or device.



Section Number:

No books, no notes, no calculators or other electronic devices. Write legibly, and show all relevant work—or risk losing credit. Answer what is asked, and only what is asked.

[30] 1. Let
$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$
.

Name:

a) Give the definition of " $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a linearly independent set" in terms of linear combinations or linear relations. Use <u>this</u> definition to show that $\mathfrak{B} = \{\vec{e}_3, \vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2\}$ is a basis of \mathbb{R}^3 .

b) Determine the \mathfrak{B} -matrix of A, where \mathfrak{B} is the basis in part a).

c) Determine all real numbers c for which the equation (for $\vec{x} \in \mathbb{R}^3$) $A\vec{x} = c\vec{x}$ has a non-trivial solution.

[20] 2. Let
$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$
.

a) Determine all vectors \vec{x} that satisfy the linear system $A\vec{x} = \vec{0}$.

b) Explain why $W = \{\vec{b} \in \mathbb{R}^4 : A\vec{x} = \vec{b} \text{ is consistent} \}$ is a *subspace* of \mathbb{R}^4 .

c) Let W be as in part b). Show that $\vec{b}_1 = 3\vec{e}_1 + 2\vec{e}_2 + \vec{e}_3 - \vec{e}_4 \in W$.

d) Let $\vec{b_1}$ be as in part c). Let $\vec{x_1}$ be **one** vector that satisfies $A\vec{x_1} = \vec{b_1}$. Explain why the solution set of $A\vec{x} = \vec{b_1}$ equals the set of all vectors of the following form: $\vec{x_1}$ plus a solution of the equation in part a).

[20] 3. a) Let $A = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$. Determine a diagonal matrix D with real entries, and an orthogonal matrix S for which $A = SDS^{-1}$. Multiply out SDS^{-1} to check that your answer is correct.

b) Let $B = \begin{bmatrix} 4 & 3 \\ -3 & -4 \end{bmatrix}$. Determine whether B is similar to the matrix A (from part a) in $\mathbb{R}^{2 \times 2}$.

- [20] 4. Let P_4 be, as usual, the linear space of polynomials of degree ≤ 4 .
 - a) Specify an isomorphism $\Phi: P_4 \to \mathbb{R}^n$ for some n. Explain why it is an isomorphism.
 - b) True or false: Every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. **Explain.**
- c) True or false: There exists an inner product on P_4 with the property that $\{1, t, t^2, t^3, t^4\}$ is an orthonormal set. **Explain.**
- d) (Convince yourself that $\mathfrak{A} = \{2, 3-t, 4t-t^2, 5-t^3, 6-t^4\}$ is a basis of P_4 .) Determine the \mathfrak{A} -coordinate vector of t^4+t .
- [20] 5. a) Let V be a linear space. Suppose that λ is an eigenvalue of the linear transformation $T:V\to V$. Derive the fact that λ^2 is an eigenvalue of T^2 .
- b) Determine all matrices in $\mathbb{R}^{3\times3}$ that are both symmetric and orthogonal, and describe them geometrically. [Suggestion: Express the two conditions in terms of "transpose."]

[20] 6. Determine the matrix (for the standard basis of \mathbb{R}^5) of the orthogonal projection of \mathbb{R}^5 onto the "plane" with equations $x_1 - x_5 = 0$, $x_1 + x_2 + x_3 + x_4 = 0$.

[20] 7. For which $a \in \mathbb{R}$, $b \in \mathbb{R}$ does the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ b & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$ have an eigenbasis (for \mathbb{R}^3)? When it does, specify an eigenbasis (depending on a and b).

[5,5,10] 8. Let V be $Span\{1, \sin x, \cos x\}$. The dimension of V is 3. We define a linear transformation $T: V \to \mathbb{R}^4$ by $T(f) = 2f(0)\vec{e}_1 + f(\pi)\vec{e}_2 - f(2\pi)\vec{e}_3$.

a) Determine the rank of T.

b) Determine a basis for the kernel of T.

c) Let D denote the linear operator on V given by D(f) = f'. Determine the complex eigenvalues of D—that includes the real ones!—and the corresponding eigenspaces.

[10] 9. Let $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$, where b and c are real scalars. Determine the set of values of b and c for which the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is asymptotically stable (meaning: for all initial states, the state vector tends to $\vec{0}$ as $t \to \infty$).

[10] 10. Determine whether $x_1^2 + 3x_1x_2 + 2x_2^2 = 1$ is the equation of an ellipse.

[10] 11. a) Give an example of two 2×2 real matrices that have the same characteristic polynomial yet they are not similar. **Explain.**

b) True or False: If a matrix fails to diagonalize over \mathbb{R} , it will diagonalize over \mathbb{C} . Explain.