Solutions to Questions for Midterm III review from Final Fall 06

3. (a) Let $A = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$. Determine a diagonal matrix D, and an orthogonal matrix S for which $A = SDS^{-1}$. Multiply out SDS^{-1} to check that your answer is correct.
(b) Let $B = \begin{bmatrix} 4 & 3 \\ -3 & -4 \end{bmatrix}$. Determine whether B is similar to the matrix A from part (a) in $\mathbb{R}^{2\times 2}$.
Sol. (a) The characteristic polynomial is $(4-\lambda)(-4-\lambda) - 9 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$. We have $\text{Ker}\,(A+5I) = \text{Span}\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$ and $\text{Ker}\,(A-5I) = \text{Span}\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\}$.
Let $S = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$.
(b) The characteristic polynomial is $(4-\lambda)(-4-\lambda) + 9 = \lambda^2 - 7 = (\lambda - \sqrt{7})(\lambda + \sqrt{7})$. Since similar matrices have the same eigenvalues B can not be similar to A.

5. (a) Let V be a linear space. Suppose that λ is an eigenvalue of the linear transformation $T : V \to V$. Derive the fact that λ^2 is an eigenvalue of T^2.
(b) Determine all matrices in $\mathbb{R}^{3\times 3}$ that are both symmetric and orthogonal, and describe them geometrically. [Suggestion: Express the two conditions in terms of 'transpose'.]
Sol. (a) Since λ is an eigenvalue there is a $\mathbf{v} \neq \mathbf{0}$ such that $T\mathbf{v} = \lambda\mathbf{v}$. Hence $T^2\mathbf{v} = T(T(\mathbf{v})) = T(\lambda\mathbf{v}) = \lambda T(\mathbf{v}) = \lambda^2\mathbf{v}$ which proves that λ^2 is an eigenvalue for T^2.
(b) $A^T A = I$ and $A^T = A$ so $A^2 = I$. Moreover A is diagonalizable so $A = QDQ^T$, where D is diagonal and $Q^T Q = QQ^T = I$. Hence $A^2 = QDQ^T DQ^T = QD^2Q^T = I$ so $D^2 = Q^T I Q = I$. It follows that the eigenvalues of A are all -1 or 1. On the other if D is diagonal with ± 1 in the diagonals then $A = QDQ^T$ then $A^2 = Q^T D^2 Q = Q^T I Q = I$.

7. For which $a \in \mathbb{R}$, $b \in \mathbb{R}$ does the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ b & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$ have an eigenbasis (for $\mathbb{R}^3$)?
When it does, specify an eigenbasis (depending on a and b).
Sol. Since the matrix is triangular the eigenvalues are the diagonal elements 1 and 2.
$(A - I)\mathbf{x} = \mathbf{0}$ is equivalent to $x_1 = 0$ and $ax_2 = 0$.
Hence if $a \neq 0$ $\text{Ker}\,(A - I) = \text{Span}\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$, and if $a = 0$ $\text{Ker}\,(A - I) = \text{Span}\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$.
$(A - 2I)\mathbf{x} = \mathbf{0}$ is equivalent to $bx_1 - x_2 = 0$ and $ax_2 - x_3 = 0$.
$\text{Ker}\,(A - 2I) = \text{Span}\{\begin{bmatrix} 1 \\ b \\ ab \end{bmatrix}\}$.
Hence A has an eigenbasis only if $a = 0$.

8. Let V be Span$\{1, \sin x, \cos x\}$. The dimension of V is 3.
(c) Let D denote the linear operator on V given by $D(f) = f'$. Determine the complex eigenvalues of D-that includes the real ones!-and the corresponding eigenspaces.

Sol. The matrix is $A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. The characteristic polynomial is $\lambda(\lambda^2+1)=\lambda(\lambda+i)(\lambda-i)$.

$\text{Ker}\,\big(A - iI\big) = \text{Span}\{\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}\}$, $\text{Ker}\,\big(A + iI\big) = \text{Span}\{\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}\}$ and $\text{Ker}\,\big(A - 0I\big) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$,
so the eigenvectors are $i \sin x + \cos x$ and $-i \sin x + \cos x$ and 1.

9. Let $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$, where b and c are real scalars. Determine the set of values of b and c for which the dynamical system $\mathbf{x}(t+1) = A\mathbf{x}(t)$ is asymptotically stable
(meaning: for all initial states, the state vector tends to $\mathbf{0}$, as $t \to \infty$.)
Sol. The characteristic polynomial is $(1-\lambda)^2 - bc = (\lambda - 1 - \sqrt{bc})(\lambda - 1 + \sqrt{bc})$.
If $bc > 0$ the eigenvalues are $\lambda = 1 \pm \sqrt{bc}$, if $bc < 0$ then $\lambda = 1 \pm i\sqrt{|bc|}$ and if $bc = 0$ $\lambda = 1$.
If $bc \neq 0$ the eigenvalues are distinct and therefore we have a basis of eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$.
If $bc \neq 0$ we can therefore write $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$. It follows that $\mathbf{x}(k) = A^k\mathbf{x}(0) = c_1A^k\mathbf{b}_1 + c_2A^k\mathbf{b}_2 = c_1\lambda_1^k\mathbf{b}_1 + c_2\lambda_2^k\mathbf{b}_2$. Hence $\mathbf{x}(k) \to 0$ as $k \to \infty$ only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
If $bc \neq 0$ at least one eigenvalue satisfy $|\lambda| \geq 1$ so it is not asymptotically stable.
If $b = c = 0$ the matrix is the identity so the eigenvalues are both 0 and it is not stable.
If $c = 0$ but $b \neq 0$ (or the other way around) then we have at least one eigenvector $\mathbf{b}_1$ with eigenvalue $\lambda_1 = 1$ so if the solution initially is in the state i.e. $\mathbf{x}(0) = c_1\mathbf{b}_1$, with $c_1 \neq 0$ then $\mathbf{x}(k) = c_1\mathbf{b}_1$, for all k which does not tend to 0 as $k \to \infty$. Hence the system is not stable.
Rem If $A=\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ then we do not have basis of eigenvectors so we can not use this method.
It is, however, easy to see that $A^k = \begin{bmatrix} 1 & kb \\ 0 & 1 \end{bmatrix}$.

10. Determine whether $q(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2 = 1$ is the equation of an ellipse.
Sol. $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$, where $A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}$. The characteristic polynomial is $(1-\lambda)(2-\lambda) - 9/4 = \lambda^2 - 3\lambda + 2 - 9/4 = (\lambda - 3/2)^2 - 5/2$, so the eigenvalues are $\lambda_1 = 3/2 - \sqrt{5/2} < 0$ and $\lambda_2 = 3/2 + \sqrt{5/2} > 0$. Since A is symmetric we can diagonalize $A = QDQ^T$ and we get $q(\mathbf{x}) = \langle \mathbf{x}, QDQ^T\mathbf{x} \rangle = \langle Q^T\mathbf{x}, DQ^T\mathbf{x} \rangle = \langle \mathbf{y}, D\mathbf{y} \rangle = \widetilde{q}(\mathbf{y})$, where $\mathbf{y} = Q^T\mathbf{x}$. Hence $q(\mathbf{x}) = \widetilde{q}(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$ is not an ellipse in the $\mathbf{y}$ coordinates, which is just a rotation or reflection of the $\mathbf{x}$ coordinates.

11. (a) Give an example of a 2×2 real matrices that have the same characteristic polynomial yet they are not similar. Explain.
(b) True or False: If a matrix fails to diagonalize over $\mathbb{R}$, it will diagonalize over $\mathbb{C}$. Explain.
Sol. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (b) False, e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$