Solutions to Math 201 Final Fall 06

1. Let $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & -1 & -2 \end{bmatrix}$.

(a) Give the definition of '$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a *linearly independent* set' **in terms of linear combinations or linear relations**. Use this definition to show that $\mathcal{B} = \{\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1-\mathbf{e}_2\}$ is a *basis* of $\mathbb{R}^3$.

(b) Determine the $\mathcal{B}$-matrix of A where $\mathcal{B}$ is the basis in part (a).

(c) Determine all real numbers c for which the equation (for $\mathbf{x}\in\mathbb{R}^3$) $A\mathbf{x}=c\mathbf{x}$ has a non-trivial solution.

Sol. (a) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if and only if the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

is $c_1=c_2=\cdots=c_k=0$. If

$$c_1\mathbf{e}_3 + c_2(\mathbf{e}_1 + \mathbf{e}_2) + c_3(\mathbf{e}_1 - \mathbf{e}_2) = (c_2 + c_3)\mathbf{e}_1 + (c_2 - c_3)\mathbf{e}_2 + c_1\mathbf{e}_3 = \mathbf{0}$$

then since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent we must have $c_2+c_3 = c_2-c_3 = c_1 = 0$, and it is easy to see that this implies that $c_1=c_2=c_3=0$ which implies that $\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1-\mathbf{e}_2$ are linearly independent. Since it is three vectors in a three dimensional space this implies that they form a basis.

(b) A determines a linear transformation $T(\mathbf{x}) = A\mathbf{x}$. If we express $\mathbf{x}$ and $T(\mathbf{x})$ in the basis $\mathbf{b}_1 = \mathbf{e}_3$, $\mathbf{b}_2 = \mathbf{e}_1+\mathbf{e}_2$, $\mathbf{b}_3 = \mathbf{e}_1-\mathbf{e}_2$, then the linear transformation of their coefficients;

$$\begin{array}{ccc} \mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 & \xrightarrow{A} & T(\mathbf{x}) = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + d_3\mathbf{b}_3 \\ \downarrow & & \downarrow \\ [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \end{array}$$

has matrix B such that $\mathbf{d} = B\mathbf{c}$. Substituting $\mathbf{b}_i$ in for $\mathbf{x}$ we see that

$$B = \begin{bmatrix} | & | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & [T(\mathbf{b}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$$

We have $T(\mathbf{b}_1)=\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}=2\mathbf{b}_2-2\mathbf{b}_1$, $T(\mathbf{b}_2)=\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}=4\mathbf{b}_2+\mathbf{b}_3-\mathbf{b}_1$ and $T(\mathbf{b}_3)=\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}=-\mathbf{b}_2+\mathbf{b}_1$

so $B = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. One sees that $T\big(\mathbf{b}_1+2\mathbf{b}_3\big) = \mathbf{0}$ so 0 is an eigenvalue.

$$(c) \quad \begin{vmatrix} 2-\lambda & 3 & 2 \\ 1 & 2-\lambda & 2 \\ 0 & -1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & -\lambda \\ 1 & 1-\lambda & -\lambda \\ 0 & -1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & \lambda & \lambda \\ 1 & 1-\lambda & -\lambda \\ 0 & -1 & -2-\lambda \end{vmatrix} = \lambda \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1-\lambda & -\lambda \\ 0 & -1 & -2-\lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2-\lambda & 1-\lambda \\ 0 & -1 & -2-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 1-\lambda \\ -1 & -2-\lambda \end{vmatrix} = -\lambda\big(\lambda^2 - \lambda - 3\big) = -\lambda\big((\lambda - 1/2)^2 - 13/4\big).$$

Hence the eigenvalues are 0 and $1/2 \pm \sqrt{21/4}$.

2. Let $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$

(a) Determine all vectors $\mathbf{x}$ that satisfy the linear system $A\mathbf{x} = \mathbf{0}$.
(b) Explain why $W = \{\mathbf{b} \in \mathbb{R}^4;\ A\mathbf{x} = \mathbf{b} \text{ is consistent}\}$ is a *subspace* of $\mathbb{R}^4$.
(c) Let W be the set in part (b). Show that $\mathbf{b}_1 = 3\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 \in W$.
(d) Let $\mathbf{b}_1$ be as in part (c). Let $\mathbf{x}_1$ be **one** vector that satisfies $A\mathbf{x}_1 = \mathbf{b}_1$. Explain why the solution set of $A\mathbf{x} = \mathbf{b}_1$ equals the set of all vectors of the following form: $\mathbf{x}_1$ plus a solution of the equation in part (a).

Sol. (a) Row reduction on the augmented matrix gives

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(b) We need to show that $\mathbf{b} \in W$ and $\mathbf{d} \in W$ then $\mathbf{b} + \mathbf{d} \in W$ and $c\mathbf{b} \in W$ for any scaler c. If $\mathbf{b} \in W$ and $\mathbf{d} \in W$ then $\mathbf{b} = A\mathbf{x}$ and $\mathbf{d} = A\mathbf{y}$ for some $\mathbf{x}$ and $\mathbf{y}$. Hence $\mathbf{b} + \mathbf{d} = A\mathbf{x} + A\mathbf{y} = A(\mathbf{x} + \mathbf{y})$ so $\mathbf{b} + \mathbf{d} \in W$. Similarly, if $\mathbf{b} \in W$ then $c\mathbf{b} = cA\mathbf{x} = A(c\mathbf{x})$ so $c\mathbf{b} \in W$.
(c) $\mathbf{b}_1$ is seen to thre be the second column of A so it is equal to $A\mathbf{e}_2$.
(d) Let $\mathbf{x}$ be any solution to $A\mathbf{x} = \mathbf{b}_1$. Then $A(\mathbf{x} - \mathbf{x}_1) = A\mathbf{x} - A\mathbf{x}_1 = \mathbf{b}_1 - \mathbf{b}_1 = \mathbf{0}$ and hence $\mathbf{x} - \mathbf{x}_1$ is a solution to the linear system in (a).

3. (a) Let $A = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$. Determine a diagonal matrix D, and an orthogonal matrix S for which $A = SDS^{-1}$. Multiply out SDS^{-1} to check that your answer is correct.
(b) Let $B = \begin{bmatrix} 4 & 3 \\ -3 & -4 \end{bmatrix}$. Determine whether B is similar to the matrix A from part (a) in $\mathbb{R}^{2\times 2}$.

Sol. (a) The characteristic polynomial is $(4-\lambda)(-4-\lambda) - 9 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$. We have $\operatorname{Ker}\big(A + 5I\big) = \operatorname{Span}\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$ and $\operatorname{Ker}\big(A - 5I\big) = \operatorname{Span}\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\}$.

Let $S = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$.

(b) The characteristic polynomial is $(4-\lambda)(-4-\lambda) + 9 = \lambda^2 - 7 = (\lambda - \sqrt{7})(\lambda + \sqrt{7})$. Since similar matrices have the same eigenvalues B can not be similar to A.

4. Let P_4 be, as usual, the linear space of polynomials of degree ≤ 4.
(a) Specify an isomorphism $\Phi : P_4 \to \mathbb{R}^n$, for some n. Explain why it is an isomorphism.
(b) *True or False:* Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. **Explain**.
(c) *True or False:* There exists an inner product on P_4 with the property that $\{1, t, t^2, t^3, t^4\}$ is an orthonormal set. **Explain**.
(d) (Convince yourself that $\mathcal{U} = \{2, 3-t, 4t-t^2, 5-t^3, 6-t^4\}$ is a basis for P_4.) Determine the $\mathcal{U}$-coordinate vector of $t^4 + t$.
Sol. An isomorphism is an invertible linear transformation.
(a) Let Φ be the linear transformation taking polynomials $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ to $(a_0, a_1, a_2, a_3, a_4) \in \mathbb{R}^5$. It is onto since one can pick any coefficients and it is one to one since the zero polynomial is the only polynomial with all the coefficients equal to 0.
(b) False, T has to be invertible.
(c) True, because one can compose with the isomorphism in (a) and take the standard inner product in $\mathbb{R}^5$.
(d) It is a basis because the set in (c) is a basis and one can obtain it from that set. $t^4 + t = -(6-t^4) + 3{\cdot}2 - (3-t) + (3/2)2$.

5. (a) Let V be a linear space. Suppose that λ is an eigenvalue of the linear transformation $T : V \to V$. Derive the fact that λ^2 is an eigenvalue of T^2.
(b) Determine all matrices in $\mathbb{R}^{3\times 3}$ that are both symmetric and orthogonal, and describe them geometrically. [Suggestion: Express the two conditions in terms of 'transpose'.]
Sol. (a) Since λ is an eigenvalue there is a $\mathbf{v} \neq \mathbf{0}$ such that $T\mathbf{v} = \lambda\mathbf{v}$. Hence $T^2\mathbf{v} = T(T(\mathbf{v})) = T(\lambda\mathbf{v}) = \lambda T(\mathbf{v}) = \lambda^2\mathbf{v}$ which proves that λ^2 is an eigenvalue for T^2.
(b) $A^TA = I$ and $A^T = A$ so $A^2 = I$. Moreover A is diagonalizable so $A = QDQ^T$, where D is diagonal and $Q^TQ = QQ^T = I$. Hence $A^2 = QDQ^TDQ^T = QD^2Q^T = I$ so $D^2 = Q^TIQ = I$. It follows that the eigenvalues of A are all -1 or 1. On the other if D is diagonal with ± 1 in the diagonals then $A = QDQ^T$ then $A^2 = Q^TD^2Q = Q^TIQ = I$.

6. Determine the matrix (for the standard basis of $\mathbb{R}^5$) of the orthogonal projection of $\mathbb{R}^5$ onto the 'plane' with equations $x_1 - x_5 = 0$, $x_1 + x_2 + x_3 + x_4 = 0$.

Sol. We need to find an orthonormal set of vectors spanning the intersection of the two hyper-planes $W = \{\mathbf{x} \in \mathbb{R}^5;\ x_1 - x_5 = 0, \text{ and } x_1 + x_2 + x_3 + x_4 = 0\}$. First we find 3 linearly independent vectors. x_3, x_4 and x_5 are free variables so the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Next we can use Gram-Schmidt to construct an orthonormal set from these three vectors. However, one can be a bit clever about it to do less work. The first and second vector above lie in the subset $W_1 = \{(0, x_2, x_3, x_4, 0) \in \mathbb{R}^5;\ x_2 + x_3 + x_4 = 0\} \subset W$. It is easy two find two orthogonal vectors in the three dimensional subset $W_1' = \{(x_2, x_3, x_4) \in \mathbb{R}^3;\ x_2 + x_3 + x_4 = 0\}$.

In fact we can take $\mathbf{v}_1' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2' = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and then we get two orthogonal vectors in W by setting the first and last component to be 0. Additionally we need one vector in $(x_1, x_2, x_3, x_4, x_5) \in W$ that is orthogonal to W_1, and since the first and second component in W_1 is zero this means the vector (x_2, x_3, x_4) has to be orthogonal to the plane W_1', and hence be proportional to the normal $(1, 1, 1)$ to this plane. Moreover we must have $x_1 = x_5$ so it must be of the form $(x_1, 1, 1, 1, x_5)$ but for this to lie in W we must have $x_1 + 1 + 1 + 1 = 0$ and hence must be $(-3, 1, 1, 1, -3)$.

Hence we get three orthogonal vectors by $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$, and we get three orthonormal vectors by $\mathbf{u}_1 = \mathbf{v}_1/\|\mathbf{v}_1\|$, $\mathbf{u}_2 = \mathbf{v}_2/\|\mathbf{v}_2\|$, $\mathbf{u}_3 = \mathbf{v}_3/\|\mathbf{v}_3\|$.

The projection is given by $\text{Proj}_W(\mathbf{x}) = (\mathbf{u}_1 \cdot \mathbf{x})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{x})\mathbf{u}_2 + (\mathbf{u}_3 \cdot \mathbf{x})\mathbf{u}_3$. This can be written as a matrix product $\text{Proj}_W(\mathbf{x}) = \mathbf{u}_1\mathbf{u}_1^T\mathbf{x} + \mathbf{u}_2\mathbf{u}_2^T\mathbf{x} + \mathbf{u}_3\mathbf{u}_3^T\mathbf{x} = \big(\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \mathbf{u}_3\mathbf{u}_3^T\big)\mathbf{x}$. Hence the matrix for the projection is $P = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \mathbf{u}_3\mathbf{u}_3^T = QQ^T$, where $Q = \begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ | & | & | \end{bmatrix}$.

7. For which $a \in \mathbb{R}$, $b \in \mathbb{R}$ does the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ b & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$ have an eigenbasis (for $\mathbb{R}^3$)?

When it does, specify an eigenbasis (depending on a and b).

Sol. Since the matrix is triangular the eigenvalues are the diagonal elements 1 and 2.

$(A - I)\mathbf{x} = \mathbf{0}$ is equivalent to $x_1 = 0$ and $ax_2 = 0$.

Hence if $a \neq 0$ $\text{Ker}\,(A - I) = \text{Span}\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$, and if $a = 0$ $\text{Ker}\,(A - I) = \text{Span}\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$.

$(A - 2I)\mathbf{x} = \mathbf{0}$ is equivalent to $bx_1 - x_2 = 0$ and $ax_2 - x_3 = 0$.

$\text{Ker}\,(A - 2I) = \text{Span}\{\begin{bmatrix} 1 \\ b \\ ab \end{bmatrix}\}$. Hence A has an eigenbasis only if $a = 0$.

8. Let V be Span$\{1, \sin x, \cos x\}$. The dimension of V is 3.
(c) Let D denote the linear operator on V given by $D(f) = f'$. Determine the complex eigenvalues of D-that includes the real ones!-and the corresponding eigenspaces.

Sol. The matrix is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. The characteristic polynomial is $\lambda(\lambda^2+1) = \lambda(\lambda+i)(\lambda-i)$.
$\text{Ker}\,\big(A - iI\big) = \text{Span}\{\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}\}$, $\text{Ker}\,\big(A + iI\big) = \text{Span}\{\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}\}$ and $\text{Ker}\,\big(A - 0I\big) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$,
so the eigenvectors are $i\sin x + \cos x$ and $-i\sin x + \cos x$ and 1.

9. Let $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$, where b and c are real scalars. Determine the set of values of b and c for which the dynamical system $\mathbf{x}(t+1) = A\mathbf{x}(t)$ is asymptotically stable
(meaning: for all initial states, the state vector tends to $\mathbf{0}$, as $t \to \infty$.)
Sol. The characteristic polynomial is $(1-\lambda)^2 - bc = (\lambda - 1 - \sqrt{bc})(\lambda - 1 + \sqrt{bc})$.
If $bc > 0$ the eigenvalues are $\lambda = 1 \pm \sqrt{bc}$, if $bc < 0$ then $\lambda = 1 \pm i\sqrt{|bc|}$ and if $bc = 0$ $\lambda = 1$.
If $bc \neq 0$ the eigenvalues are distinct and therefore we have a basis of eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$.
If $bc \neq 0$ we can therefore write $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$. It follows that $\mathbf{x}(k) = A^k\mathbf{x}(0) = c_1 A^k\mathbf{b}_1 + c_2 A^k\mathbf{b}_2 = c_1\lambda_1^k\mathbf{b}_1 + c_2\lambda_2^k\mathbf{b}_2$. Hence $\mathbf{x}(k) \to 0$ as $k \to \infty$ only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
If $bc \neq 0$ at least one eigenvalue satisfy $|\lambda| \geq 1$ so it is not asymptotically stable.
If $b = c = 0$ the matrix is the identity so the eigenvalues are both 0 and it is not stable.
If $c = 0$ but $b \neq 0$ (or the other way around) then we have at least one eigenvector $\mathbf{b}_1$ with eigenvalue $\lambda_1 = 1$ so if the solution initially is in the state i.e. $\mathbf{x}(0) = c_1\mathbf{b}_1$, with $c_1 \neq 0$ then $\mathbf{x}(k) = c_1\mathbf{b}_1$, for all k which does not tend to 0 as $k \to \infty$. Hence the system is not stable.
Rem If $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ then we do not have basis of eigenvectors so we can not use this method.
It is, however, easy to see that $A^k = \begin{bmatrix} 1 & kb \\ 0 & 1 \end{bmatrix}$.

10. Determine whether $q(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2 = 1$ is the equation of an ellipse.
Sol. $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$, where $A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}$. The characteristic polynomial is $(1-\lambda)(2-\lambda) - 9/4 = \lambda^2 - 3\lambda + 2 - 9/4 = (\lambda - 3/2)^2 - 5/2$, so the eigenvalues are $\lambda_1 = 3/2 - \sqrt{5/2} < 0$ and $\lambda_2 = 3/2 + \sqrt{5/2} > 0$. Since A is symmetric we can diagonalize $A = QDQ^T$ and we get $q(\mathbf{x}) = \langle \mathbf{x}, QDQ^T\mathbf{x} \rangle = \langle Q^T\mathbf{x}, DQ^T\mathbf{x} \rangle = \langle \mathbf{y}, D\mathbf{y} \rangle = \widetilde{q}(\mathbf{y})$, where $\mathbf{y} = Q^T\mathbf{x}$. Hence $q(\mathbf{x}) = \widetilde{q}(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$ is not an ellipse in the $\mathbf{y}$ coordinates, which is just a rotation or reflection of the $\mathbf{x}$ coordinates.

11. (a) Give an example of a 2×2 real matrices that have the same characteristic polynomial yet they are not similar. Explain.
(b) True or False: If a matrix fails to diagonalize over $\mathbb{R}$, it will diagonalize over $\mathbb{C}$. Explain.
Sol. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (b) False, e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$