Solutions to Questions for Midterm III Review from Final Spring 05

5a. [2 marks] Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

5b. [3 marks] Give a factorization $A = QDQ^T$ where Q has orthonormal columns and D is a diagonal matrix.

5d. [1 marks] Is A a positive definite matrix? Why? Give the quadratic form $q(x, y)$ associated to A.

<u>Sol.</u>

5a. Solve $\det(A - \lambda I) = 0$. We get the equation $\lambda(\lambda - 5) = 0$. Hence $\lambda_1 = 5$ and $\lambda_2 = 0$.

5b. $A = QDQ^T$ is an "eigenvalue-eigenvector" factorization of a symmetric matrix. D is a diagonal matrix containing the eigenvalues of A and Q is a 2×2-matrix whose orthonormal columns are eigenvectors of A. For example

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

5d. A is not a positive definite matrix as $\lambda_2 = 0$. The quadratic form (singular) associated to A is $q(x, y) = x^2 + 4xy + 4y^2$.

6a. [3 marks] If possible, find an invertible matrix M such that

$$M^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

If it is not possible, state why M cannot exist.

6b. [3 marks] For what real values of c (if any) is $A = \begin{bmatrix} -1 & c & 2 \\ c & -4 & -3 \\ 2 & -3 & 4 \end{bmatrix}$ a symmetric positive definite matrix?

6c. [4 marks] Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$. Is the quadratic form $q(x, y)$ associated to A positive definite? Find its principal axes.

<u>Sol.</u>

6a. Not possible. The condition means that $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ is similar to $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Similar matrices have equal traces and rank. But $\text{trace}(A) = 3 \neq \text{trace}(B) = 5$ and $\text{rk}(A) = 1 \neq \text{rk}(B) = 2$.

6b. Not possible. For symmetric positive-definite matrices all upper-left determinants are greater than zero. Note that the 1 by 1 upper-left determinant is -1.

6c. $\det(A) < 0$, hence A (or equivalently its associated quadratic form $q(x, y) = 3x^2 + 8xy + 3y^2$) is not positive definite. The eigenvalues of A are: $\lambda_1 = -1$ and $\lambda_2 = 7$. The principal axes are the eigenspaces of A, namely $E_1 = \text{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ and $E_2 = \text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

7a. [6 marks] Let $A_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Is A_1 diagonalizable? Why? Is A_1 invertible? Why? Determine the spectral decomposition of A_1 into projection matrices.

7b. [3 marks] Let $A_2 = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$. Is A_2 invertible? Why? Is A_2 diagonalizable? Why? Determine (if exists) a matrix S and a diagonal matrix D such that $S^{-1}A_2S = D$.

7c. [6 marks] Describe the linear transformation $T_{A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ associated to A_2. Does such A_2 have a decomposition into projection matrices? If yes, give it.

Sol.

7a. A_1 is symmetric, hence A_1 is diagonalizable. A_1 is invertible as $\det(A_1) = \prod \lambda_i = 2 \cdot 1 \cdot (-1) = -2 \neq 0$. The spectral decomposition of A_1 is given by

$$A_1 = \sum_{i=1}^{3} \lambda_i \underline{x}_i \underline{x}_i^T$$

where $\underline{x}_i$ are eigenvectors associated to the eigenvalues λ_i. We can choose $\underline{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as eigenvector associated to $\lambda_1 = 2$, $\underline{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ as eigenvector associated to $\lambda_2 = 1$ and $\underline{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as eigenvector associated to $\lambda_3 = -1$. It follows that the spectral decomposition of A_1 is

$$A_1 = 2P_1 + P_2 - P_3 = 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7b. $\det(A_2) = 0$ hence A_2 is not invertible. The eigenvalues of A_2 are $\mu_1 = 0$ and $\mu_2 = -4$. They are distinct, hence A_2 is diagonalizable. The columns of the matrix S are made by 2 eigenvectors of A i.e. $S = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$, whereas $D = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ is the eigenvalues matrix.

7c. The linear transformation $T_{A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ is determined by a projection P_2 onto the line spanned by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{0}, \qquad A_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

It follows that

$$A_2 = -4 \begin{bmatrix} 3/4 & -3/4 \\ -1/4 & 1/4 \end{bmatrix} = -4P_2$$

is the required decomposition, where P_2 is a projection matrix.

8a. [3 marks] Find the lengths and the inner product $\underline{x} \cdot \underline{y}$ of the following complex vectors

$$\underline{x} = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix}, \qquad \underline{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad (i^2 = -1).$$

8b. [3 marks] Let $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$. Let $\underline{x}_1$, $\underline{x}_2$ be two (linearly independent) eigenvectors of A. Compute $\underline{x}_1 \cdot \underline{x}_2$ and show that $\det(A) \in \mathbb{R}$.

8c. [4 marks] Prove that for any complex vector $\underline{x}$

$$\underline{x}^H A \underline{x} \in \mathbb{R}. \qquad (H = \text{Hermitian})$$

Sol.

8a. $\text{length}(\underline{x}) = (\underline{x}^H \underline{x})^{1/2} = (\begin{bmatrix} 2+4i & -4i \end{bmatrix} \cdot \begin{bmatrix} 2-4i \\ 4i \end{bmatrix})^{1/2} = 6$; $\text{length}(\underline{y}) = (\underline{y}^T \underline{y})^{1/2} = \sqrt{20}$ and $\underline{x} \cdot \underline{y} := \underline{x}^H \underline{y} = 4(1-2i)$.

8b. Notice that $A = A^H$, furthermore let λ_i be the 2 eigenvalues of A: $0 = \lambda_1 \neq \lambda_2 = 3$, then $\underline{x}_1 \cdot \underline{x}_2 = 0$. Also, one knows that every eigenvalue of a Hermitian matrix is real and so will be its determinant ($\det(A) = \lambda_1 \lambda_2 = 0$).

8c. We have $(\underline{x}^H A \underline{x})^H = \underline{x}^H A \underline{x}$, as $A = A^H$. It follows that $\underline{x}^H A \underline{x} \in \mathbb{R}$.