Solutions to Questions for Midterm III Review from Final Spring 06

11. Find all eigenvalues of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

Sol. Since A is triangular the eigenvalues are the diagonal entries $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.

12. For each of the eigenvalues of A, find the associated eigenspace.

Sol. For $\lambda_1 = \lambda_2 = 0$ we get the system $(A - 0I)\mathbf{x} = \mathbf{0}$, which is equivalent to $\mathbf{x}_3 = 0$ and $\mathbf{x}_2 = 0$ and hence the eigenspace is one dimensional $\text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.

For $\lambda_3 = 2$ we get the system $(A - 2I)\mathbf{x} = \mathbf{0}$, which is equivalent to $-2\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_2 = 0$ and $-\mathbf{x}_2 + 2\mathbf{x}_3 = 0$ and hence the eigenspace is one dimensional $\text{Span}\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\}$.

13. Is it possible to diagonalize the matrix A?

Sol. A is not diagonalizable since it does not have three linearly independent eigenvectors.

19. True or False: If A and B are both symmetric matrices, them their product AB must also be symmetric. Explain the reasoning behind your answer.

Sol. False, take e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

20. True or False: If A and B are both orthogonal matrices, them their product AB must also be orthogonal. Explain the reasoning behind your answer.

Sol. True, since A and B are orthogonal $A^TA = I$ and $B^TB = I$, and it follows that $(AB)^TAB = B^TA^TAB = B^TIB = B^TB = I$ so AB is orthogonal.

21. How many complex eigenvalues does the matrix $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 3 & 3 \\ 1 & 2 & 3 & 0 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$ have?

Sol. Since M is symmetric all the eigenvalues are real.

22. Express the quadratic form $q(x_1, x_2) = x_1^2 + 6x_1x_2 + 8x_2^2$ as an inner product $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x}\rangle$, where A is a symmetric matrix.

Sol. $A = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}$.

23. Is there a choice of numbers (x_1, x_2) for which $q(x_1, x_2)$ is negative? What does the set of points where $q(x_1, x_2) = 1$ look like? [Please describe the overall shape of the set - it is not necessary to give exact specifications.]

Sol. The characteristic polynomial is $(1-\lambda)(8-\lambda) - 9 = \lambda^2 - 9\lambda - 1$ which has roots $\lambda_\pm = 9/2 \pm \sqrt{(9/2)^2 + 1}$ so $\lambda_- < 0$ and $\lambda_+ > 0$. Since A is symmetric it can be diagonalized $A = QDQ^T$ and if we set $\mathbf{y} = Q^T\mathbf{x}$ we get

$$q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, QDQ^T\mathbf{x}\rangle = \langle Q^T\mathbf{x}, DQ^T\mathbf{x}\rangle = \langle \mathbf{y}, D\mathbf{y}\rangle = \lambda_- y_1^2 + \lambda_+ y_2^2 = \widetilde{q}(\mathbf{y}).$$

Hence $\widetilde{q}(1,0) = \lambda_- < 0$. Now, $\mathbf{y}_0 = (y_1, y_2) = (1, 0)$ corresponds to some $\mathbf{x}_0 = Q\mathbf{y}_0$ such that $q(\mathbf{x}_0) = \widetilde{q}(\mathbf{y}_0) = \lambda_- < 0$. The set $\widetilde{q}(\mathbf{y}) = \lambda_- y_1^2 + \lambda_+ y_2^2 = 1$ is a hyperbola.

24. What are the singular values of matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$?

Sol. The singular values are the square root of the eigenvalues of $A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

The characteristic polynomial is $(6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 9/2)^2 - 25/4$, so the eigenvalues are $9/2 \pm 5/2$ so $\lambda_1 = 7$ and $\lambda_2 = 2$ and the singular values are $\sigma_1 = \sqrt{7}$ and $\sigma_2 = \sqrt{2}$.

25. Find a set of perpendicular vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^2$ which have the additional property that $A\mathbf{v}_1$ and $A\mathbf{v}_2$ are also perpendicular to each other?

Sol. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the normalized eigenvectors of A^TA:

$(A^TA - 7I)\mathbf{v}_1 = 0$ gives $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(A^TA - 2I)\mathbf{v}_2 = 0$ gives $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

We claim that $A\mathbf{v}_1$ and $A\mathbf{v}_2$ are perpendicular. In fact,

$$\langle A\mathbf{v}_i, A\mathbf{v}_j\rangle = \langle A^TA\mathbf{v}_i, \mathbf{v}_j\rangle = \langle \lambda_i\mathbf{v}_i, \mathbf{v}_j\rangle = \lambda_i\langle \mathbf{v}_i, \mathbf{v}_j\rangle,$$

and if $i \neq j$ then $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$.

Remark If $i = j$ the above equation reads $\|A\mathbf{v}_i\|^2 = \lambda_i\|A\mathbf{v}_i\|^2$, so the vectors $\mathbf{u}_i = A\mathbf{v}_i/\sigma_i$, $i = 1, 2$, are orthonormal. We have $\mathbf{u}_1 = \frac{1}{\sqrt{35}}\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ and $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

The vectors can be used to obtain the singular value decomposition $A = U\Sigma V^T$, where

$$V = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix},\ U = \begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{35}} & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{-2}{\sqrt{14}} \\ \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$