Solutions to Math 201 Final Spring 06

1. How many solutions are there to the system of linear equation $\begin{cases} x_1 - 3x_2 = 0 \\ 3x_1 - 2x_2 = 7 \\ 2x_1 + x_2 = 7 \end{cases}$

Sol. The second equation is the sum of the first and third so it is not needed. The remaining 2×2 system $\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$ has an invertible coefficient matrix so it is uniquely solvable.

2. The vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ form a basis for $\mathbb{R}^2$. Express the vector $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Sol. Since $\mathbf{e}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ gives the system $2c_1 + 5c_2 = 1$ and $c_1 + 3c_2 = 0$. The second equation gives $c_1 = -3c_2$ which if we substitute into the first gives $-c_2 = 1$ so $c_2 = -1$ and $c_1 = 3$.

3. Suppose a linear transformation T has the property that $T(\mathbf{v}_1) = \mathbf{v}_1 + \mathbf{v}_2$ and $T(\mathbf{v}_2) = 2\mathbf{v}_1 + 3\mathbf{v}_2$. Use your answer to problem 2 to find the value of $T(\mathbf{e}_1)$.

Sol. $T(\mathbf{e}_1) = T\big(3\mathbf{v}_1 - \mathbf{v}_2\big) = 3(\mathbf{v}_1 + \mathbf{v}_2) - \big(2\mathbf{v}_1 + 3\mathbf{v}_2\big) = \mathbf{v}_1$.

4. Are the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ all linearly independent? If not, identify which of these vectors are redundant.

Sol. They are linearly dependent since it is four vectors in a three dimensional space. The rest of the problem is most easily solved as in Problem 5.

5. Consider the matrix $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ -1 & -2 & 1 & -3 \\ 1 & 2 & 2 & 0 \end{bmatrix}$. Choose a basis for the image of A.

Sol. Row reduction gives

$$\begin{bmatrix} 2 & 4 & 3 & 1 \\ -1 & -2 & 1 & -3 \\ 1 & 2 & 2 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 \\ 1 & 2 & 2 & 0 \end{bmatrix} \begin{matrix} (1)-2(3) \\ (2)+(3) \\ \, \end{matrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (3) \\ -(1) \\ (2)+3(1) \end{matrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (1)-2(2) \\ \, \\ \, \end{matrix}.$$

The first and third column of $B = \text{RREF}(A)$ corresponds to the leading variables and as a consequence the first and third column of A form a basis for the image of A, see section 3.2-3.

Rem. This is because any linear relation amongst the columns of B corresponds to the same relation amongst the columns of A since $B\mathbf{x} = \mathbf{0}$ has the same solution set as $A\mathbf{x} = \mathbf{0}$, i.e. if $A = [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]$, $B = [\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4]$ then $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{0}$ if and only if $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + x_4\mathbf{b}_4 = \mathbf{0}$. We have $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$ so $\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$ and $\mathbf{b}_2 = 2\mathbf{b}_1$ so $\mathbf{a}_2 = 2\mathbf{a}_1$.

6. Chose a basis for the kernel of A.

Sol. By the solution to problem 5, x_2 and x_4 are free variables so the and $x_1 + 2x_2 + x_4 = 0$ and $x_3 - x_4 = 0$ so the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and these vectors form a basis for the kernel.

7. Let $Tf(x) = \dfrac{f(x) - f(0)}{x}$, acting on functions f.
If a domain of T is $\mathcal{P}_n = \{\text{polynomials } f(x) = a_0 + a_1 x + \cdots = a_n x^n\}$,
then what is the image of T?
Sol. The image is polynomial of degree one less $Tf = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$.

8. Show that the kernel of T is the space $\mathcal{P}_0$ of constant functions.
Sol. $Tf = 0$ is equivalent to $f = a_0$.

9. What is the determinant of the matrix $M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$?

Sol. Subtracting a multiple of the first row from the second and third rows gives

$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 5 \\ 1 & 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1 \cdot 5 \cdot 3 \cdot 2 = 30,$$

since the determinant of a triangular matrix is the product of the diagonal elements.
Rem. Note that we only used the row operators that subtract off a multiple of another row from the row we change and these don't change the determinant.
Note also that alternatively as a second step one could have expanded along the first column.

10. Give an example of 2×2 matrices A and B where $\det(A) + \det(B)$ is not equal to $\det(A + B)$.
Sol. If $A = I$ and $B = -I$.

11. Find all eigenvalues of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

Sol. Since A is triangular the eigenvalues are the diagonal entries $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.

12. For each of the eigenvalues of A, find the associated eigenspace.
Sol. For $\lambda_1 = \lambda_2 = 0$ we get the system $(A - 0I)\mathbf{x} = \mathbf{0}$, which is equivalent to $\mathbf{x}_3 = 0$ and $\mathbf{x}_2 = 0$ and hence the eigenspace is one dimensional $\text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.
For $\lambda_3 = 2$ we get the system $(A - 2I)\mathbf{x} = \mathbf{0}$, which is equivalent to $-2\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_2 = 0$ and $-\mathbf{x}_2 + 2\mathbf{x}_3 = 0$ and hence the eigenspace is one dimensional $\text{Span}\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\}$.

13. Is it possible to diagonalize the matrix A?
Sol. A is not diagonalizable since it does not have three linearly independent eigenvectors.

14. What is the length of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$?

Sol. $\|\mathbf{v}\| = \sqrt{4}$.

15. What is the angle between the vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$?

Sol. $\|\mathbf{v}\| \, \|\mathbf{e}_1\| \cos\theta = \mathbf{v} \cdot e_2 = 1$ so $\cos\theta = 1/\sqrt{4}$.

16. What is the projection of $\mathbf{e}_2$ onto the line spanned by $\mathbf{v}$?

Sol. Let $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$. Then the projection is $(\mathbf{e_2} \cdot \mathbf{u})\mathbf{u} = \dfrac{\mathbf{e_2} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \dfrac{1}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

17. The system of equations $\begin{cases} x_1 = 5 \\ x_1 = 1 \\ x_1 = 6 \end{cases}$ $\left(\text{equivalently, } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_1 = \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}\right)$ is hopelessly inconsistent. What values of x_1 provides the least-squares approximate solution?

18. Decide whether the function $\langle f, g \rangle = \displaystyle\int_{-1}^{1} f(x)g(-x)dx$ is a valid inner product, where f and g are allowed to be any pair of continuous functions on the interval $[-1, 1]$.
Sol It is not since we can take $f(t) = 0$ when $t < 0$ but $f(t) = t$ for $t > 0$ in which case $\langle f, f \rangle = 0$ but $f \neq 0$.

19. True or False: If A and B are both symmetric matrices, them their product AB must also be symmetric. Explain the reasoning behind your answer.
Sol. False, take e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

20. True or False: If A and B are both orthogonal matrices, them their product AB must also be orthogonal. Explain the reasoning behind your answer.
Sol. True, since A and B are orthogonal $A^T A = I$ and $B^T B = I$, and it follows that $(AB)^T AB = B^T A^T AB = B^T IB = B^T B = I$ so AB is orthogonal.

21. How many complex eigenvalues does the matrix $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 3 & 3 \\ 1 & 2 & 3 & 0 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$ have?

Sol. Since M is symmetric all the eigenvalues are real.

22. Express the quadratic form $q(x_1, x_2) = x_1^2 + 6x_1x_2 + 8x_2^2$ as an inner product $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$, where A is a symmetric matrix.
Sol. $A = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}$.

23. Is there a choice of numbers (x_1, x_2) for which $q(x_1, x_2)$ is negative? What does the set of points where $q(x_1, x_2) = 1$ look like? [Please describe the overall shape of the set - it is not necessary to give exact specifications.]

Sol. The characteristic polynomial is $(1-\lambda)(8-\lambda)-9=\lambda^2-9\lambda-1$ which has roots $\lambda_\pm = 9/2 \pm \sqrt{(9/2)^2+1}$ so $\lambda_- < 0$ and $\lambda_+ > 0$. Since A is symmetric it can be diagonalized $A = QDQ^T$ and if we set $\mathbf{y} = Q^T\mathbf{x}$ we get

$$q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, QDQ^T\mathbf{x}\rangle = \langle Q^T\mathbf{x}, DQ^T\mathbf{x}\rangle = \langle \mathbf{y}, D\mathbf{y}\rangle = \lambda_- y_1^2 + \lambda_+ y_2^2 = \widetilde{q}(\mathbf{y}).$$

Hence $\widetilde{q}(1,0) = \lambda_- < 0$. Now, $\mathbf{y}_0 = (y_1, y_2) = (1,0)$ corresponds to some $\mathbf{x}_0 = Q\mathbf{y}_0$ such that $q(\mathbf{x}_0) = \widetilde{q}(\mathbf{y}_0) = \lambda_- < 0$. The set $\widetilde{q}(\mathbf{y}) = \lambda_- y_1^2 + \lambda_+ y_2^2 = 1$ is a hyperbola.

24. What are the singular values of matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$?

Sol. The singular values are the square root of the eigenvalues of $A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

The characteristic polynomial is $(6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 9/2)^2 - 25/4$, so the eigenvalues are $9/2 \pm 5/2$ so $\lambda_1 = 7$ and $\lambda_2 = 2$ and the singular values are $\sigma_1 = \sqrt{7}$ and $\sigma_2 = \sqrt{2}$.

25. Find a set of perpendicular vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^2$ which have the additional property that $A\mathbf{v}_1$ and $A\mathbf{v}_2$ are also perpendicular to each other?

Sol. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the normalized eigenvectors of A^TA:

$(A^TA - 7I)\mathbf{v}_1 = 0$ gives $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(A^TA - 2I)\mathbf{v}_2 = 0$ gives $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

We claim that $A\mathbf{v}_1$ and $A\mathbf{v}_2$ are perpendicular. In fact,

$$\langle A\mathbf{v}_i, A\mathbf{v}_j\rangle = \langle A^TA\mathbf{v}_i, \mathbf{v}_j\rangle = \langle \lambda_i\mathbf{v}_i, \mathbf{v}_j\rangle = \lambda_i\langle \mathbf{v}_i, \mathbf{v}_j\rangle,$$

and if $i \neq j$ then $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$.

Remark If $i = j$ the above equation reads $\|A\mathbf{v}_i\|^2 = \lambda_i \|A\mathbf{v}_i\|^2$, so the vectors $\mathbf{u}_i = A\mathbf{v}_i/\sigma_i$, $i = 1, 2$, are orthonormal. We have $\mathbf{u}_1 = \frac{1}{\sqrt{35}}\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ and $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

The vectors can be used to obtain the singular value decomposition $A = U\Sigma V^T$, where

$$V = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \quad U = \begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{35}} & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{-2}{\sqrt{14}} \\ \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$