Math 201 **Name (Print):** ____________________

Spring 2014

Final

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Lecturer: Jesus Martinez Garcia

Time Limit: 3 hours Section: ____________________

This exam contains 10 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a theorem of lemma you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Write with blue or black pen only**. We need to keep the exams for a year and pencil would fade out.
- **The last question is a bonus question**. You do not need to attempt it. It can only increase your grade.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	40	
3	15	
4	10	
5	15	
6	10	
Total:	110	

Do not write in the table to the right.

1. (20 points) **Inner product spaces.**

   (a) (5 points) Let V be a vector space. Give the definition of an inner product on V.

   Solution: $\langle , \rangle \colon V \times V \to \mathbb{R}$ is an inner product if $\forall f, g, h \in V$ and $\forall c \in \mathbb{R}$, the following rules are satisfied:

   1. $\langle f, g\rangle = \langle g, f\rangle$.
   2. $\langle cf, g\rangle = c\langle f, g\rangle$.
   3. $\langle f, f\rangle \geqslant 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0_V$.
   4. $\langle f + h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$.

   (b) (5 points) Let $C^\infty(0,1)$ be the set of differentiable functions $f \colon (0,1) \to \mathbb{R}$. Show that

$$\langle f, g\rangle = \int_0^1 \Big(f(x)g(x) + f'(x)g'(x) \Big) dx$$

   is an inner product on V.

   Solution: $\langle , \rangle$ satisfies the properties of the previous exercise. Indeed

   1. $$\langle f, g\rangle = \int_0^1 \big(f(x)g(x) + f'(x)g'(x) \big)\, dx = \langle g, f\rangle.$$

   2. $$\langle cf, g\rangle = \int_0^1 \big(cf(x)g(x) + cf'(x)g'(x) \big)\, dx = c\int_0^1 \big(f(x)g(x) + f'(x)g'(x) \big)\, dx = c\langle f, g\rangle.$$

   3. $$\langle f, f\rangle = \int_0^1 \big(f(x)^2 + (f'(x))^2 \big)\, dx = \leqslant 0 \text{ and } = 0 \text{ if and only if } f(x) = 0,\ \forall x \in (0,1).$$

   4. $$\begin{aligned}\langle f + h, g\rangle &= \int_0^1 \Big(\big(f(x) + h(x)\big)\, g(x) + \big(f'(x) + h'(x)\big)\, g'(x) \Big)\, dx = \\ &= \int_0^1 \big(f(x)g(x) + f'(x)g'(x) \big)\, dx + \int_0^1 \big(h(x)g(x) + h'(x)g'(x) \big)\, dx = \\ &= \langle f, g\rangle + \langle h, g\rangle.\end{aligned}$$

(c) (10 points) Compute **all** the Fourier coefficients of $f(t) = t^2$.

Solution: Recall tha the Fourier coefficients of $f(t) = t^2$ are the numbers a_0, b_k, c_k for $k \in \mathbb{N}_{>0}$ such that

$$\mathrm{proj}_{T_n} t^2 = a_0 \frac{1}{\sqrt{2}} + b_1 \sin t + c_1 \cos t + \cdots + b_n \sin(nt) + c_n \cos(nt),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{t^2}{\sqrt{2}} dt = \left. \frac{t^3}{3\sqrt{2}\pi} \right|_{-\pi}^{\pi} = \frac{2}{\sqrt{2}3} \pi^2,$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \sin(kt) dt = 0,$$

since $t^2 \sin(kt)$ is an odd function for all k and, integrating by parts twice:

$$\begin{aligned}
c_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos(kt) dt \\
&= \frac{1}{\pi} \left(\cancelto{0}{\left. \frac{t^2}{k} (\sin(kt)) \right|_{-\pi}^{\pi}} - \frac{1}{k} \int_{-\pi}^{\pi} 2t \sin(kt)\, dt \right) \\
&= \frac{-2}{k\pi} \int_{-\pi}^{\pi} \sin(kt) t \; dt \\
&= \frac{2}{k\pi} \left(\left. \frac{t}{k} (cos(kt)) \right|_{-\pi}^{\pi} - \cancelto{0}{\frac{1}{k} \int_{-\pi}^{\pi} \cos(kt) dt} \right) \\
&= \frac{4}{k^2} (-1)^k.
\end{aligned}$$

2. (40 points) **Bases, Image, Kernel and dimension.** Please, note that the following exercises can be attempted independently. You may assume previous parts even if you have not completed them (e.g. you can assume part (a), T is linear, when solving part (d), even if you do not prove that T is linear).

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R})$ and $T\colon \mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}_2(\mathbb{R})$ be defined by $T(M) = AM - MA$.

(a) (5 points) Show that T is a linear transformation

Solution: T is linear, since

1. $$\begin{aligned} T(M+N) &= A(M+N) - (M+N)A = AM + AN - MA - NA \\ &= (AM - MA) + (AN - NA) = T(M) + T(N). \end{aligned}$$

2. $$T(\lambda M) = A(\lambda M) - (\lambda M)A = \lambda(AM - MA) = \lambda T(M).$$

(b) (5 points) Show that

$$\mathfrak{B} = \left\{ v_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ v_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $\mathrm{Mat}_2(\mathbb{R})$.

Solution: We check the elements in $\mathfrak{B}$ are linearly independent. Suppose that we have $c_1, \ldots, c_4$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \begin{pmatrix} c_2 + c_3 + c_4 & c_4 \\ c_1 + c_3 & c_1 + c_2 \end{pmatrix}.$$

Then $c_4 = 0, c_1 = -c_3, c_2 = -c_1 = c_3$ and $0 = c_2 + c_3 + c_4 = 2c_3$, so $c_1 = c_2 = c_3 = c_4 = 0$ and the elements of $\mathfrak{B}$ are linearly independent. Since $\langle \mathfrak{B} \rangle \subset \mathrm{Mat}_2(\mathbb{R})$, which is a space of dimension 4, and $\mathfrak{B}$ has 4 linearly independent elements, it must span $\mathrm{Mat}_2(\mathbb{R})$, and therefore $\mathfrak{B}$ is a basis of $\mathrm{Mat}_2(\mathbb{R})$.

(c) (10 points) Let

$$\mathfrak{E} = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be another basis of $\mathrm{Mat}_2(\mathbb{R})$. Compute the change of basis matrix $S = S_{\mathfrak{B} \to \mathfrak{E}}$

Solution: Observe $v_1 = e_3 + e_4$, $v_2 = e_1 + e_4$, $v_3 = e_1 + e_3$, $v_4 = e_1 + e_2$, and therefore

$$S_{\mathfrak{B} \to \mathfrak{E}} = \begin{pmatrix} | & | & | & | \\ [v_1]_{\mathfrak{E}} & [v_2]_{\mathfrak{E}} & [v_3]_{\mathfrak{E}} & [v_4]_{\mathfrak{E}} \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

(d) (10 points) Find B, the $\mathfrak{B}$-matrix of T.

Solution: The inverse of S above is given by

$$S^{-1} = S_{\mathfrak{E}\to\mathfrak{B}} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

If E is the $\mathfrak{E}$-matrix of T, then the $\mathfrak{B}$-matrix B of T is given by $B = S^{-1}ES$. Therefore, we compute E first:

$$\begin{aligned}
T(e_1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -e_2 \\
T(e_2) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
T(e_3) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e_1 - e_4 \\
T(e_4) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_2.
\end{aligned}$$

Therefore, the $\mathfrak{E}$-matrix of T is given by

$$E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and the $\mathfrak{B}$-matrix is

$$\begin{aligned}
B &= S^{-1}ES = \\
&= \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \\
&= \begin{pmatrix} \frac{-1}{2} & 0 & \frac{-3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{3}{2} & \frac{1}{2} \\ -1 & 0 & -1 & -1 \end{pmatrix}.
\end{aligned}$$

(e) (5 points) Give a basis of $\mathrm{Im}(T)$. What is the dimension of $\mathrm{Im}(T)$?

Solution: From the matrix E, we deduce that

$$\mathrm{Im}(T) = \langle -e_2, e_1 - e_4, e_2 \rangle = \langle e_2, e_1 - e_4 \rangle.$$

These vectors are linearly independent. Indeed:

$$\overrightarrow{0} = a\overrightarrow{e_2} + b(\overrightarrow{e_1} - \overrightarrow{e_4}) = b\overrightarrow{e_1} + a\overrightarrow{e_2} - b\overrightarrow{e_4}.$$

but $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_4}$ are linearly independent, hence $a = b = 0$, and $\{\overrightarrow{e_1} - \overrightarrow{e_4}, \overrightarrow{e_2}\}$ is a basis of $\mathrm{Im}(T)$ and $\dim(\mathrm{Im}(T)) = 2$.

(f) (5 points) Give a basis of $\mathrm{Ker}(T)$. What is the dimension of $\mathrm{Ker}(T)$?

Solution: We know that $\dim(\mathrm{Ker}(T)) = 4 - \dim(\mathrm{Im}(T)) = 2$ by the Rank-Nullity Theorem. Therefore it is enough to find two linearly independent vectors in $\mathrm{Ker}(T)$. Let $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathrm{Ker}(T)$. Then:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = E \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

This implies that $c = 0$ and $a = d$. Hence, all vectors in $\mathrm{Ker}(T)$ are $\begin{pmatrix} a \\ b \\ 0 \\ a \end{pmatrix}$. Two linearly independent vectors are:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Therefore, a basis of $\mathrm{Ker}(T)$ is $\{e_1 - e_4, e_2\}$.

3. (15 points) **True/False.**

   Decide if the following statements are True or False. If they are true, provide a proof. If they are false, provide a counter-example.

   (a) (5 points) Let A be an invertible matrix. Then $(A^2)^{-1} = (A^{-1})^2$.
   Solution: True, since

   $$(A^2) \cdot (A^{-1})^2 = A(AA^{-1})A^{-1} = A \cdot A^{-1} = I.$$

   (b) (5 points) Let $A, B \in \mathrm{Mat}_2(\mathbb{R})$. If $AB = 0$, then $BA = 0$.
   Solution: False. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

   $$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   (c) (5 points) The space P_7 (polynomials of degree at most 7) is isomorphic to $\mathrm{Mat}_{2\times 4}(\mathbb{R})$.
   Solution: True, since

   $$P_7 = \{a_0 + a_1 x + \cdots + a_7 x^7 | a_0, \ldots, a_7 \in \mathbb{R}\} \cong \mathbb{R}^8$$

   and $\mathrm{Mat}_{2\times 4}(\mathbb{R}) \cong \mathbb{R}^8$, therefore

   $$P_7 \cong \mathbb{R}^8 \cong \mathrm{Mat}_{2\times 4}(\mathbb{R}).$$

4. (10 points) **Determinants and orthogonal matrices.**

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Find A^{-1} by computing the adjoint of A.

Solution: By Sarrus' rule:

$$\det A = 3 + 0 + 2 - 1 - 0 - 0 = 4.$$

Then $A^{-1} = \frac{1}{\det A}\mathrm{Adj}(A)$. If we let $B = \mathrm{Adj}(A) = (b_{ij})$ we have $b_{ij} = (-1)^{i+j}\det(A_{ji})$, and

$$\begin{aligned} b_{11} &= 3, & b_{12} &= -6, & b_{13} &= 1, \\ b_{21} &= 1, & b_{22} &= 2, & b_{23} &= -1, \\ b_{31} &= -1, & b_{32} &= 2, & b_{33} &= 1. \end{aligned}$$

Hence

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -6 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{pmatrix}.$$

5. (15 points) Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\vec{x}) = A\vec{x}$. Find an **orthonormal** basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are orthogonal. What are the norms of $T(\vec{v}_1)$ and $T(\vec{v}_2)$?

Solution: As we saw in the course, we must find an orthonormal eigenbasis $\{\vec{v_1}, \vec{v_2}\}$ of $A^T A$. Then, the vectors $\{A\vec{v_1}, A\vec{v_2}\}$ will be orthogonal, but not necessarily normal and their lengths will be the singular values $\|A\vec{v_1}\| = \sigma_1 = \sqrt{\lambda_1}$, $\|A\vec{v_1}\| = \sigma_2 = \sqrt{\lambda_2}$, where λ_1, λ_2 are eigenvalues of $A^T A$.

$$B := A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues are given by the characteristic polynomial

$$f_B(\lambda) = \det(B - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 3 = 0.$$

The roots of this polynomial are $\lambda_1 = 3, \lambda_2 = 1$. The eigenspaces are

$$E_3 = \operatorname{Ker}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle,$$

$$E_1 = \operatorname{Ker}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle.$$

Therefore, normalising these generating vectors we obtain an orthonormal eigenbasis of R^2, given by

$$\vec{v_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The image by A of this basis consists of two orthogonal vectors of lengths

$$\|A\vec{v_1}\| = \sqrt{3}, \text{ and } \|A\vec{v_1}\| = 2.$$

6. (10 points) **BONUS QUESTION:** Find the matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ of the orthogonal projection of $\mathbb{R}^n$ onto the line spanned by the vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

(Hint: Try first for $n = 3$).

Solution: Let $\overrightarrow{v} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $L = \langle \overrightarrow{v} \rangle$. Let $\overrightarrow{w} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $\overrightarrow{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.. Then $\|\overrightarrow{w}\| = 1$ and $\mathrm{proj}_L \overrightarrow{x} = (\overrightarrow{x} \cdot \overrightarrow{w})\overrightarrow{w} = \frac{x_1 + \cdots x_n}{\sqrt{n}} \frac{1}{\sqrt{n}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Hence $\mathrm{proj}_L \overrightarrow{x} = A \cdot \overrightarrow{x}$, where $A = \frac{1}{n} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$.