Linear algebra (math 110.201)

Final Exam - 4 May 2016

Name: ____________________

Section number/TA: ____________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **3 hours**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; **points will be subtracted otherwise**.
(7) Once you submit your exam, you will not be allowed to modify it.
(8) By submitting this exam, you are agreeing to the above terms. Cheating or refusing to stop writing when time is called may result in an automatic failure.

Exercise	Points	Your score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
18	10	
Total	180	

Exercise 1. *Consider the following system of equations, where λ is a real number:*

$$\begin{aligned} 2W + 2X + 3Y + 3Z &= 1 \\ 3W + 3X + 4Y + 4Z &= \lambda^2 \\ 4W + 4X + 5Y + 5Z &= \lambda \end{aligned}$$

(1) *(2 points) Write down the augmented matrix of this system.*
(2) *(4 points) For which real numbers λ does this system have at least one solution? Can it ever have exactly one solution?*
(3) *(4 points) For the cases in which the system has at least one solution, describe all solutions.*

Solution:

Exercise 2. *(10 points) Let a denote a real number, and consider the matrix:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ a & 0 & a & 1 \end{pmatrix}$$

For which values of a is this matrix invertible? For those values of a, write down A^{-1} in terms of a.

Solution:

Exercise 3. *Consider the following vectors:*

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \vec{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

(1) *(4 points) Calculate* $||\vec{x}||$ *and* $||\vec{y}||$.
(2) *(4 points) Calculate the dot product* $\vec{x} \cdot (\vec{x} - \vec{y})$.
(3) *(2 points) Calculate the distance betweeen* $\vec{x}$ *and* $\vec{y}$.

Solution:

Exercise 4. *(10 points) Suppose that $\vec{x}$ and $\vec{y}$ are linearly independent vectors in $\mathbb{R}^n$. Show that the vectors $\vec{x} + \vec{y}$ and $\vec{x} - \vec{y}$ must also be linearly independent.*

Solution:

Exercise 5. *Consider the vectors* $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -3 \end{bmatrix}$ *and* $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$. *Let* $W \subseteq \mathbb{R}^4$ *denote the set of all vectors in* $\mathbb{R}^4$ *which are orthogonal to both* $\vec{x}$ *and* $\vec{y}$*.*

(1) *(5 points) Verify that* W *is a subspace of* $\mathbb{R}^4$*.*
(2) *(5 points) Find a basis for* W*. What is* $\dim(W)$*?*

Solution:

Exercise 6. *Consider the following matrix:*

$$A = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{smallmatrix}\right)$$

(1) *(2 points) Compute* $\mathrm{RREF}(A)$.
(2) *(4 points) Find a basis for* $\mathrm{Ker}(A)$. *What is* $\dim(\mathrm{Ker}(A))$*?*
(3) *(4 points) Find a basis for* $\mathrm{Im}(A)$. *What is* $\dim(\mathrm{Im}(A))$*?*

Solution:

Exercise 7. *(10 points) Let $\mathscr{B}$ denote the basis $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$ of $\mathbb{R}^2$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose matrix with respect to $\mathscr{B}$ is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. What is the matrix of T with respect to the standard basis of $\mathbb{R}^2$? Express your answer in terms of a, b, c, d.*

Solution:

Exercise 8. *(10 points) Let B be a $m \times n$ matrix and let C be a $n \times k$ matrix. Show that if $\vec{y}$ is in $\mathrm{Im}(BC)$, then $\vec{y}$ is in $\mathrm{Im}(B)$. Explain why this tells us that rank$(BC) \leq$ rank(B).*

Solution:

Exercise 9. *Let* $W \subseteq P_3(\mathbb{R})$ *denote the set of all polynomials of dergree* ≤ 3 *which satisfy* $f(1) = 0$ *and* $f(-2) = 0$.

(1) *(2 points) Give an example of a polynomial of degree exactly equal to* 3 *which satisfies* $f(1) = 0$ *and* $f(-2) = 0$.
(2) *(4 points) Show that* W *is a subspace of* $P_3(\mathbb{R})$.
(3) *(4 points) Find a basis for* W. *What is* $\dim(W)$*?*

Solution:

Exercise 10. *Consider the following vectors in* $\mathbb{R}^5$*:*

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 12 \\ 9 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and put $W = Span(v_1, v_2, v_3)$*. These vectors are linearly independent.*

(1) *(4 points) Perform the Gram-Schmidt process on* v_1, v_2, v_3 *to obtain an orthonormal basis of* W*.*
(2) *(4 points) What is the matrix of* $\mathrm{Proj}_W : \mathbb{R}^5 \to \mathbb{R}^5$*?*
(3) *(2 points) Calculate the projection of the vector*

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

onto W*.*

Solution:

Exercise 11. *(10 points) Consider the following matrix and vector:*

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{pmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The system of equations $A\vec{x} = \vec{y}$ has no exact solutions $\vec{x}$ in $\mathbb{R}^3$. Find all least squares solutions to this system.

Solution:

Exercise 12. *Consider the vector space* $\mathcal{C}[0,1]$ *of continuous functions on the interval* $[0,1]$, *together with its inner product* $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

(1) *(5 points) Viewing the space* $P_1(\mathbb{R})$ *of polynomials of degree* ≤ 1 *as a subspace of* $\mathcal{C}[0,1]$, *construct an orthonormal basis of* $P_1(\mathbb{R})$ *from the basis* $1, X$ *using the Gram-Schmidt algorithm.*

(2) *(5 points) Let* f *denote the function* $f(t) = t^2$ *on* $[0,1]$. *Calculate* $\mathrm{Proj}_{P_1(\mathbb{R})}(f)$.

Solution:

Exercise 13. *(10 points) Suppose that A is an $n \times n$ matrix having the property that the sum of entries in each row is equal to 0. For example, the following matrix has this property:*

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & -2 & -1 \\ -5 & 1 & 4 \end{pmatrix}$$

If A is an $n \times n$ matrix with this property, what must $\det(A)$ be equal to?

Solution:

Exercise 14. *(10 points) Consider the following matrix:*

$$A = \begin{pmatrix} 0 & 1 & 0 & -2 & 0 & 2 \\ 0 & 2 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 5 & -3 & 0 & 2 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 1 & 3 & 0 & 1 & 0 & 2 \end{pmatrix}$$

(1) *Calculate* $\det(A)$.

(2) *Is the matrix* A *invertible?*

Solution:

Exercise 15. *(10 points) Suppose A is a $n \times n$ real matrix with $A^7 = I_n$. What is* $\det(A)$*?*

Solution:

Exercise 16. *Consider the following matrix:*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}$$

(1) *(5 points) Which of the following vectors are eigenvectors of A? For those which are eigenvectors, state what their corresponding eigenvalues are.*

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{x}_4 = \begin{bmatrix} 7 \\ 12 \\ 11 \end{bmatrix}$$

(2) *(5 points) Is A diagonalizable? Explain why or why not. (You are not asked to diagonalize A).*

Solution:

Exercise 17. *Consider the following matrix:*

$$\begin{pmatrix} -15 & 56 \\ -4 & 15 \end{pmatrix}$$

(1) *(2 points) Find all real eigenvalues of A.*
(2) *(6 points) Is A diagonalizable? If so, write it in the form $A = S\Lambda S^{-1}$ with Λ diagonal and S invertible.*
(3) *(2 points) Compute A^{200}.*

Solution:

Exercise 18. *Consider the matrix*

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

The matrix A has 1 *and* 2 *as eigenvalues.*

(1) *(3 points) Find a basis for the eigenspace E_1.*
(2) *(3 points) Find a basis for the eigenspace E_2.*
(3) *(2 points) Can A have any other eigenvalues? Why or why not?*
(4) *(2 points) Is A diagonalizable? Explain why or why not. (You are not asked to diagonalize A).*