1. Lecture 1: 1.1 Linear systems of equations

In Linear Algebra we solve **systems of linear equations**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where the a_{ij}'s and b_i's are given **constants** and $x_1, x_2, ..., x_n$ are **unknowns** to be determined. A **solution** to the $m \times n$ **system** is an ordered n-tuple $(x_1, x_2, ..., x_n)$ such that all the m equations are satisfied. The set of all solutions are called the **solution set**.

Linear Algebra is a set of concepts and techniques related to understanding and solving linear systems.

Note that the text book has the convention that m and n are switched in the above definition.

Geometric interpretation of Solution set

Let us try to understand the geometric meaning of a general 2×2 **systems**:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

The solutions to each of the equations form a line in the (x_1, x_2)-plane.
(x_1, x_2) is therefore a solution to the system if and only if it lies on both these lines.

Ex 1 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{cases}$

Sol The two lines in the plane intersect at the point $(1, 2)$, which is the only solution

Ex 2 Find all solutions to the system $\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + 4x_2 = 0 \end{cases}$

Sol The lines are parallel so they don't intersect. No solutions!

Ex 3 Find all solutions to the system: $\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + 4x_2 = 6 \end{cases}$

Sol Both equations represent the same line. Every point on the line is a solution!

The same picture hold for a general system: the solution set is either empty (no solutions), or, there is exactly one solution, or there are infinitely many solutions.
A system is called **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

Analytic solution

Two systems are called **equivalent** if they have the same solution set.

Ex 4 Show that the systems are equivalent:

$$(I): \quad \begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{cases}, \qquad \Leftrightarrow \qquad (II): \quad \begin{cases} x_1 + x_2 = 3 \\ \phantom{x_1 +{}} x_2 = 2 \end{cases}$$

Sol Both systems represent the intersection of two lines that happen to intersect at the same point $(1,2)$. This can also be seen analytically, in fact if we subtract 2 times the first line of the first system from the second line of the first system we get

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 - \ x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline -3x_2 = -6 \end{array}$$

If we divide both sides by -3 we get the second equation of the first system.

Hence any solution to the first system is a solution to the second and going backwards we also see that any solution to the second system is a solution to the first.

The second system II can be solved analytically by so called **back-substitution**:
If we plug $x_2 = 2$ into the first equation we get $2x_1 - 2 = 0$, i.e. $x_1 = 1$.

To solve the first system I analytically we **reduce** it to the second system II and solve it.

I the case of a 2×2 system one could solve the second equation for x_2 in terms of x_1 and plug that into the first equation to get an equation for only x_1.
However, for bigger system one has to systematically reduce the number of variables as above.

Ex 5 Find all solutions to the system in **Ex 2**: $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 0 \end{cases}$

Sol. If we follow the same procedure above to solve this system we get into trouble

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 + 2x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline 0 = -6 \end{array} \quad \text{so we get the system} \quad \begin{array}{r} x_1 + x_2 = 3 \\ 0 = 6 \end{array}$$

which is not true. Hence we analytically found that the system in Ex 2 is inconsistent.

Triangular systems and back substitution

An $n \times n$ system (1.1) is said to be in **(upper) triangular form** if $a_{ij} = 0$, for $i > j$. The entries a_{ii} for $i = 1, ..., n$ are called the **diagonal entries**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

A triangular system with nonvanishing diagonal entries is said to be **non-degenerate**.
It is easy to solve a system in non-degenerate triangular form by **back-substitution**.

Ex Solve the system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

Sol. The last equation says that $x_3 = 3$, substituting this into the second equation gives that $x_2 = 4 + 4x_3 = 4 + 12 = 16$ and substituting both these into the first equation gives that $x_1 = 2x_2 - x_3 = 32 - 3 = 29$.

Row operations and Equivalent systems

We therefore want to transform $n \times n$ systems into equivalent triangular systems.
Let us recall what basic operations we can do that leads to **equivalent systems**:
1. A multiple of one equation may be added to another.
2. We can change the order of any two equations
3. Both sides of an equation can be multiplied by the same nonzero number.
These operations lead to systems with the same solutions since the operations are reversible.

We want to solve the 3×3 system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Geometrically this represents the intersection of 3 planes.
To minimize the writing it is convenient to only write out the **coefficient matrix**:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}, \qquad \text{and right hand side column vector} \qquad \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

or to combine them in one to the **augmented matrix** of the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \qquad \text{or just} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}.$$

Instead of doing operations on the systems one can now just do the same operations on the augmented matrix, noting that it exactly corresponds to operations on the system. This involves less writing.

Solving a 3×3 system

Ex 5 Transform the 3×3 system (written in both ways for comparison)

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

into an equivalent triangular system and solve it.

Sol We want to eliminate x_1 from the last equation by using the first:

$$\begin{array}{rc} \text{[equation 3]} & -4x_1 + 5x_2 + 9x_3 = -9 \\ +4\,\text{[equation 1]} & 4x_1 - 8x_2 + 4x_3 = 0 \\ \hline \text{[new equation 3]} & -3x_2 + 13x_3 = -9 \end{array}$$

After some practice this calculation is usually performed mentally.

Hence we get the system (written in both ways for comparison)

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \ (3) + 4(1)$$

Now first multiply the second equation by $1/2$:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \ (2)/2$$

We now want to eliminate x_2 from the last equation by adding 3 times the second:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \ (3) + 3(2)$$

Hence we got an equivalent system in non-degenerate **triangular form**.

Because the diagonal entries are nonvanishing we can solve it using **back substitution**:

$$\begin{aligned} x_1 - 2x_2 \qquad &= -3 \\ x_2 \qquad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \ \begin{array}{l} (1) - (3) \\ (2) + 4(3) \\ \end{array}$$

Having cleared up the column above x_3 in the third equation, move back to the x_2 in second equation and use it to eliminate the $-2x_2$ above it. Adding 2 times the second equation to the first gives

$$\begin{aligned} x_1 \qquad \qquad &= 29 \\ x_2 \qquad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \ \begin{array}{l} (1) + 2(2) \\ \\ \end{array}$$

Summary and Conceptual Questions

A 3×3 system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

geometrically represents the intersection of there planes.

Question What are the possible solution sets of a 3×3 system?

A system in triangular form

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 4x_3 &= 4 \\ 3x_3 &= 3 \end{aligned}$$

can be solved by back-substitution (if there is a solution)

Question How does a triangular system with no solution look like?
(Hint: Look at the diagonal elements e.g. of a diagonal system.)

We can use elementary row operations to turn an $n \times n$ system into an equivalent diagonal system.

Operations on the system corresponds to operations on the augmented matrix.
The **Elementary Row Operations** on a matrix are
1. Adding a multiple of one row to another.
2. Interchanging two rows
3. Multiplying a row by a nonzero number.
Two matrices are **row equivalent** if one can be transformed into the other by elementary row operations. Two systems have the same **solution set** if their augmented matrices are row equivalent. This is because the operations above are reversible.

Question One (or more) of the row operations above is not needed to put an $n \times n$ system in triangular form. Which?