11. Lecture 11: 3.4 Coordinates

Recall that $\mathbf{b}_1,\ldots,\mathbf{b}_n$ form a **basis** for a subspace V if they span V and are linearly independent.
The Unique Representation Theorem Let $\mathbf{b}_1,\ldots,\mathbf{b}_n$ be a basis for a subspace V. Then for any $\mathbf{x} \in V$, there is unique set of scalars $c_1,\ldots,c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n. \tag{11.1}$$

Suppose $\mathcal{B} = \{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ is a basis. The **$\mathcal{B}$-coordinates of x** are the weights $c_1,\cdots,c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ and the **$\mathcal{B}$-coordinate vector of x** is the vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Equation (11.1) can be written

$$\mathbf{x} = S\,[\mathbf{x}]_{\mathcal{B}}, \qquad \text{where} \quad S = \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix}$$

Th If $\mathcal{B} = \{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ is a basis for a subspace V then the coordinate map $\mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V to $\mathbf{R}^n$.

Ex 1 Let V be the plane $x_1 - x_2 + x_3 = 0$. Then $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ form a basis $\mathcal{B}$ for V, since they lie in the plane and are not parallel. It is easy to check that $\mathbf{x} = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$ lies in V. Find c_1 and c_2 so $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$.

Sol Using row reduction on the augmented matrix $\begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & 8 \\ 1 & -1 & 1 \end{bmatrix}$ we find that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Ex 2 Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{E}}$.

Sol We can write

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. We want to find c_1 and c_2 such that

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

solving the system gives that $c_1 = 2$ and $c_2 = 3$. Hence $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

One can graph this in standard $\mathcal{E}$ graph paper and in $\mathcal{B}$ graph paper.

Note that in the example

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

In general for a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$;

$$\mathbf{x} = S\,[\mathbf{x}]_{\mathcal{B}}\,, \qquad \text{where} \qquad S = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$

and $[\mathbf{x}]_{\mathcal{B}}$ the coordinate vector. We call S the **change-of-coordinate matrix** from the standard basis in $\mathbf{R}^n$ to the basis $\mathcal{B}$.

Ex 3 Find the coordinates of $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ in the basis $\mathbf{b}_1, \mathbf{b}_2$ in the previous example.

Sol $S = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix}$.

Then $[\mathbf{x}]_{\mathcal{B}} = S^{-1}\mathbf{x} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

Expressing a linear transformation in terms of different bases

Ex 4 Let L be the line in $\mathbf{R}^2$ that is spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
Let T be the linear transformation that projects any vector orthogonally onto L.
Find the matrix A for T in the standard coordinate system.
Sol We now pick a coordinate system $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ with $\mathbf{b}_1$ parallel to the line and $\mathbf{b}_2$ perpendicular to the line

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

If $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ then $T(\mathbf{x}) = c_1\mathbf{b}_1$. Equivalently, if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ then $[T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$:

$$[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix B for T in the $\mathcal{B}$ coordinate system is hence very simple.
The matrix for A for T in the standard coordinates is more complicated but one can calculate it from B:

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{A} & T(\mathbf{x}) \\ {\scriptstyle S}\uparrow & & \uparrow{\scriptstyle S} \\ [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} \end{array},$$

where $S = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ and $S^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$. Hence

$$A = SBS^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

In general if T is a linear transformation from $\mathbf{R}^n \to \mathbf{R}^n$ then

$$[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}$$

where

$$B = \begin{bmatrix} | & | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

Two matrices A and B are called **similar** if there is an invertible matrix S such that $A = SBS^{-1}$.

Summary and Questions

Suppose $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis for a subspace V of $\mathbf{R}^m$. The **$\mathcal{B}$-coordinates of $\mathbf{x}$** are the weights $c_1, \cdots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n,$$

We have

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = S\,[\mathbf{x}]_{\mathcal{B}}, \qquad \text{where} \qquad S = \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Here S is called the **change of coordinate matrix** and $[\mathbf{x}]_{\mathcal{B}}$ the **$\mathcal{B}$-coordinate vector of $\mathbf{x}$**.

Let T be a linear transformation from $\mathbf{R}^n \to \mathbf{R}^n$ with matrix A in the standard coordinates. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbf{R}^n$. Then there is a linear transformation, with matrix B, taking $[\mathbf{x}]_{\mathcal{B}}$ to $[T(\mathbf{x})]_{\mathcal{B}}$

$$[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}, \tag{11.2}$$

i.e. if we express $\mathbf{x}$ and $T(\mathbf{x})$ in the basis then the linear transformation of their coefficients;

$$\begin{array}{ccc} \mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n & \xrightarrow{A} & T(\mathbf{x}) = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n \\ \downarrow & & \downarrow \\ [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \end{array}$$

has matrix B. The matrix B is calculated from the following commutative diagram

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{A} & T(\mathbf{x}) \\ {\scriptstyle S}\uparrow & & \uparrow{\scriptstyle S} \\ [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} \end{array},$$

i.e. B is **similar** to A:

$$B = S^{-1}AS.$$

However, the point is that we can start from the other end and choose a basis in which T, i.e. the matrix B, becomes simple. Substituting $\mathbf{b}_i$ into (11.2) we see that

$$B = \begin{bmatrix} | & | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

If we can find a basis in which B is simple then we can find A by

$$A = SBS^{-1}.$$

In particular we may be able to find a basis in which B is diagonal, i.e, such that

$$T(\mathbf{b}_i) = \lambda_i \mathbf{b}_i, \qquad \text{for} \quad i = 1, \ldots, n.$$

This is the case for the projection onto the line L in $\mathbf{R}^2$ spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
Pick a coordinate system $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ with $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ parallel to the line and $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ perpendicular to the line. Then $T(\mathbf{b}_1) = \mathbf{b}_1$, and $T(\mathbf{b}_2) = \mathbf{0}$.