# 12. Lecture 12: 4.1 Vector Spaces

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are vectors in Euclidean space we defined the **addition** $\mathbf{x}+\mathbf{y} \in \mathbf{R}^n$ and **scalar multiplication** $\lambda\mathbf{x} \in \mathbf{R}^n$, either geometrically with arrows or algebraically in terms of coordinates. The addition and scalar multiplication satisfy certain properties listed below. These properties show up in many different contexts and these properties are what is needed to get the proofs of the theorems to go through. Rather then repeating the proofs in each new situation it is more efficient to introduce the concept of an abstract vector space to be a set with addition and scalar multiplication satisfying these properties and once and for all prove the theorems under these assumptions only.

A set $V$ with two operations, addition and multiplication by scalars, defined on it is called a **vector space** if the following properties hold for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ $\alpha, \beta \in \mathbf{R}$:

1. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$. (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w}) = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associative)
4. There is an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ all $\mathbf{u} \in V$ (additive unit)
5. For each $\mathbf{u} \in V$ there is $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (additive inverse)
6. If $\mathbf{u} \in V$ and $\alpha$ is a scalar then $\alpha\,\mathbf{u} \in V$. (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (distributive)
8. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ (distributive)
9. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ (associative)
10. $1 \cdot \mathbf{u} = \mathbf{u}$ (multiplicative unit)

It follows from (1)-(10) that $0\mathbf{u} = \mathbf{0}$, $(-1)\mathbf{u} = -\mathbf{u}$, $c\mathbf{0} = \mathbf{0}$.

**Ex 0** $\mathbf{R}^n$

**Linear space** would perhaps be a better name. In brief its a set with two operations, addition and scalar multiplication, that allows us to form linear combinations.
It is difficult to understand from the axioms what a vector space is. Instead one has to get a feeling for what it looks like by examples. If you describe to an alien that a chair is something with a seat and a back they will not understand but if you show them many chairs and how they are used they will get a good idea.

The idea of an abstract vector space goes back to Grassmann in 1844. He realized that once geometry is put into this axiomatic algebraic form it would no longer be limited to three dimensional space. Contemporary mathematicians failed to recognize the importance of his work and it was not understood until Peano in 1888 published a clear interpretation.

## Examples of vector spaces

**Ex 1** $\mathbf{R}^n$ satisfy these properties, and more generally, so do $m\times n$ matrices; $\mathbf{R}^{m\times n}$; with the sum $A+B\in\mathbf{R}^{m\times n}$ and scalar multiplication $\lambda A\in\mathbf{R}^{m\times n}$ we previously defined. $\mathbf{R}^{m\times n}\sim\mathbf{R}^{mn}$.

**Ex 2** Let $F(\mathbf{R},\mathbf{R})$ be the set of real valued functions $\mathbf{R}\to\mathbf{R}$.
If $f, g$ are functions and $\lambda$ a scalar then we can define the functions $f+g$ and $\lambda f$ by $(f+g)(t) = f(t)+g(t)$ and $(\lambda f)(t) = \lambda f(t)$. All the 10 properties above are satisfied which makes $F(\mathbf{R},\mathbf{R})$ into a vector space. A vector in $\mathbf{R}^n$ is determined by its $n$ components but to specify a function on $\mathbf{R}$ we have to give its value at infinitely many points. Still the analogy with $\mathbf{R}^n$ has proved enormously useful. E.g. to find the polynomial that best approximate a function one projects onto the closest polynomial in a certain distance.

**Ex 3** The set of infinite sequences $(x_0, x_1, x_2, \dots)$ is a vector space with addition:

$$(x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots) = (x_0+y_0, x_1+y_1, x_2+y_2, \dots)$$

## Subspaces

**Question:** When is a subset of a vector space a vector space itself?
We defined a vector spaces $V$ as a set with addition and scalar multiplication that satisfy 10 axioms. However, often we have a subset of a vector space in which case we only need to check that its closed under addition and scalar multiplication:

A subset $S$ of a vector space $V$ is called **subspace** if
(a) $\mathbf{0}\in S$, (b) $\mathbf{u}+\mathbf{v}\in S$, whenever $\mathbf{u},\mathbf{v}\in S$, (c) $\alpha\,\mathbf{u}\in S$, whenever $\mathbf{u}\in S$ and $\alpha$ is a scalar.

A subspace is automatically a vector space in its own right, i.e. with addition and scalar multiplication inherited from (coming from) $V$ it satisfies all the 10 axioms.
In fact (1) is (b) and (6) is (c). (2)-(4) and (7)-(10) hold for elements in the subspace since they are in the larger space where the axioms hold. Axiom (5), the existence of the additive inverse follows from that $-\mathbf{u}=(-1)\mathbf{u}$ is in the subspace.

**Ex 4** $P$ the space of polynomials is a subspace of $F(\mathbf{R},\mathbf{R})$.

**Ex 5** $P_N$ the space of polynomials of degree $\leq N$ is a subspace of $P$.

**Ex 6** $C^k(\mathbf{R},\mathbf{R})$, the set of $k$ times differentiable functions with continuous derivatives, is a subspace of $F(\mathbf{R},\mathbf{R})$. if $k=0$ this is written $C(\mathbf{R},\mathbf{R})$ the continuous functions.

**Ex 7** $H^k(\mathbf{R},\mathbf{R})$ $k$ times differentiable functions with the derivatives in $L^2(\mathbf{R},\mathbf{R})$. $L^2$ is the space of square integrable functions

$$\int |f(x)|^2\,dx < \infty.$$

## Basis and coordinates

$\mathcal{B} = f_1, f_2, \ldots, f_n$ is a basis for a vector space $V$ if every $f \in V$ can then be written in q unique way as

$$f = c_1 f_1 + \cdots + c_n f_n$$

then the vector $[f]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is called the coordinate vector and the map $f \to [f]_{\mathcal{B}}$ is called the coordinate map.

The coordinate mapping allows us to introduce coordinate systems for unfamiliar vector spaces:

**Ex** Standard basis for the polynomials of degree 2 or less, $\mathbf{P}_2$ is $\{1, t, t^2\}$. We can write $[a+bt+ct^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. We say that $\mathbf{P}_2$ is isomorphic to $\mathbf{R}^3$. All vector space operations in $\mathbf{P}_2$ corresponds to operations in $\mathbf{R}^3$, e.g. adding two polynomials $(-1+2t-3t^2)+(2+3t+5t^2) = 1 + 5t + 2t^2$ corresponds to adding their coordinate vectors: $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$.

**Ex** In the space of all polynomials $\mathbf{P}$ the monomials $x^n$ form a basis.

**Ex** In the spaces of sequences, $e_k = (0, \ldots, 0, 1, 0, \ldots)$, with a 1 only in $k$th space form a basis.

**Ex** In $L^2(-\pi,\pi)$ the functions $1, \cos x, \cos(2x), \ldots \sin x, \sin(2x), \ldots$ form a basis. If $f \in L^2$ then

$$f = b_0 + b_1 \cos(x) + b_2 \cos(2x) + \cdots + a_1 \sin(x) + a_2 \sin(2x) + \ldots.$$

This expansion is called the **Fourier series** and will be discussed in Chapter 5 of the book.

A point of all this is that one can approximate functions by finite polynomials or a finite series of the above form.

## Summary and Questions

A set $V$ with two operations, addition and multiplication by scalars, defined on it is called a **vector space** if the following properties hold for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ $\alpha, \beta \in \mathbf{R}$:

1. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$. (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w}) = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associative)
4. There is an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ all $\mathbf{u} \in V$ (additive unit)
5. For each $\mathbf{u} \in V$ there is $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (additive inverse)
6. If $\mathbf{u} \in V$ and $\alpha$ is a scalar then $\alpha \mathbf{u} \in V$. (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (distributive)
8. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ (distributive)
9. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ (associative)
10. $1 \cdot \mathbf{u} = \mathbf{u}$ (multiplicative unit)

**Ex 1** $\mathbf{R}^n$ satisfy these properties, and more generally, so do $m \times n$ matrices; $\mathbf{R}^{m\times n}$; with the sum $A + B \in \mathbf{R}^{m\times n}$ and scalar multiplication $\lambda A \in \mathbf{R}^{m\times n}$ we previously defined. $\mathbf{R}^{m\times n} \sim \mathbf{R}^{mn}$.

**Ex 2** Let $F(\mathbf{R}, \mathbf{R})$ be the set of real valued functions $\mathbf{R} \to \mathbf{R}$.
If $f, g$ are functions and $\lambda$ a scalar then we can define the functions $f + g$ and $\lambda f$ by $(f + g)(t) = f(t) + g(t)$ and $(\lambda f)(t) = \lambda f(t)$. All the 10 properties above are satisfied.

**Ex 3** The set of infinite sequences $(x_0, x_1, x_2, \dots)$ is a vector space with addition:
$(x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots) = (x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots)$

A subset $S$ of a vector space $V$ is called **subspace** if
(a) $\mathbf{0} \in S$, (b) $\mathbf{u} + \mathbf{v} \in S$, whenever $\mathbf{u}, \mathbf{v} \in S$, (c) $\alpha \mathbf{u} \in S$, whenever $\mathbf{u} \in S$ and $\alpha$ is a scalar.
A subspace is automatically a vector space in its own right, it satisfies all the 10 axioms.

**Ex 4** The continuous functions $C(\mathbf{R}, \mathbf{R})$ is a subspace of the space of all functions $F(\mathbf{R}, \mathbf{R})$.

**Ex 5** $P_N$ the space of polynomials of degree $\leq N$ is a subspace of all functions $F(\mathbf{R}, \mathbf{R})$.

**Ex 7** $L^2$ the space of square integrable functions $\int |f(x)|^2 \, dx < \infty$, is a subspace of $F(\mathbf{R}, \mathbf{R})$.

The 'vectors' $f_1, f_2, \dots, f_n$ form a basis for a vector space $V$ if every $f \in V$ can then be written in a unique way as $f = c_1 f_1 + \cdots + c_n f_n$

**Ex** In the space of all polynomials $\mathbf{P}$ the monomials $x^n$ form a basis.

**Ex** In the spaces of sequences, $e_k = (0, \dots, 0, 1, 0, \dots)$, with a 1 only in $k$th space form a basis.

**Ex** In $L^2(-\pi, \pi)$ the functions $1, \cos x, \cos(2x), \dots \sin x, \sin(2x), \dots$ form a basis. If $f \in L^2$ then $f = b_0 + b_1 \cos(x) + b_2 \cos(2x) + \cdots + a_1 \sin(x) + a_2 \sin(2x) + \dots$.
This expansion is called the **Fourier series** and will be discussed in Chapter 5 of the book.

One can approximate functions by finite polynomials or a finite series of the above form.