13. Lecture 13: 4.2 Isomorphisms

Let V and W be two vector spaces. Let T be a transformation from V, called the **domain**, to W, called the **target space**. Then T is called **linear** if

$$T(\mathbf{x}+\mathbf{z}) = T(\mathbf{x}) + T(\mathbf{z}), \qquad \text{and} \qquad T(\lambda \mathbf{x}) = \lambda T(\mathbf{x}).$$

The set

$$\mathrm{Im}(T) = \{T(f);\ f \in V\} \subset W$$

is called the **image** of T and the set

$$\mathrm{Ker}(T) = \{f \in V;\ T(f) = 0\} \subset V$$

is called the **kernel** of T.

If the image of $T : V \to W$ is finite dimensional then **rank** of T is $\mathrm{rank}(T) = \dim(\mathrm{Im}(T))$ and the **nullity** of T is $\mathrm{nullity}(T) = \dim(\mathrm{Ker}(T))$ and we have $\dim(V) = \mathrm{rank}(T) + \mathrm{nullity}(T)$.

T is called **onto** if $\mathrm{Im}(T) = W$.
T is called **one-to-one** if $T(f) = T(g)$ implies that $f = g$.
T is called **invertible** if T is onto and T is one-to-one.

An invertible linear transformation T is called an **isomorphism**.
We say that V and W are **isomorphic** if there is an isomorphism from V to W.
In that case T^{-1} is linear and invertible so it is an isomorphism from W to V.
If T is an isomorphism from V and W and S is an isomorphism from U and V then $T \circ S$ is an isomorphism from U and W.

Th If T is an isomorphism if and only if $\mathrm{Im}(T) = W$ and $\mathrm{Ker}(T) = \{0\}$.
Pf If $T(f) = T(g)$ then $T(f-g) = T(f) - T(g) = 0$, i.e. $f - g \in \mathrm{Ker}(T)$.
Hence if $\mathrm{Ker}(T) = \{0\}$ it follows that T is one-to-one and hence an isomorphism.
On the other hand if T is invertible then $T^{-1}(T(f)) = f$ so $T(f) = 0$ implies that $f = 0$.

If $\mathcal{B} = (f_1, \dots, f_n)$ is a basis then the coordinate transformation $f \to [f]_{\mathcal{B}}$ is an isomorphism:

$$f = c_1 f_1 + \cdots + c_n f_n \quad \to \quad [f]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Ex Consider the space $\mathbf{R}^{2\times 2}$ matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A basis $\mathcal{B}$ is given by the form matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ since we can write $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

In terms of this basis the coordinate map is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \to \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

We will consider the linear transformation which is taking the derivative
$T(f) = D(f) = f'$ from $C^\infty \to C^\infty$. (Here C^∞ are the infinitely differentiable functions.)

Ex Is D an isomorphism from P_3 to P_3? (Here P_3 are the polynomials of degree ≤ 3.) i.e. $D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$.
Sol No because it is not onto, since e.g. x^3 is not in the image.

Ex Is D an isomorphism from P_3 to P_2?
Sol No since the constants a are in the kernel.

Ex Let $P_3^0 = \{f \in P_3;\ f(0) = 0\}$, i.e. polynomials of the form $bx + cx^2 + dx^3$.
Then P_3^0 is a vector space since it is a subspace of P_3 that is closed under addition and scalar multiplication. Is $D : P_3^0 \to P_2$ an isomorphism.
Sol It is onto and the constants are no longer on the kernel since an element of P_3^0 is a polynomial of the form $bx + cx^2 + dx^3$ without the constant.
Moreover, the inverse to $D : P_3^0 \to P_2$ is $D^{-1}(f)(x) = \int_0^x f(t)\, dt$.

Ex Is $L(f) = \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix}$ an isomorphism from P_2 to $\mathbf{R}^3$?
Sol If f is in the kernel then $f(x) = c(x-1)(x-2)$ but if $f(3) = 2c = 0$ as well then $c = 0$ so $f = 0$ identically. Hence $\text{Ker}(L) = \{0\}$ so $\text{nullity}(L) = 0$. Since dimension of P_2 is 3 we have that $\text{nullity}(L) + \text{rank}(L) = 3$ it follows that $\text{rank}(L) = 3$, which is the same as the dimension of $\mathbf{R}^3$. Therefore $\text{Im}(T) = \mathbf{R}^3$ (since it is a subspace and the only 3 dimensional subspace of $\mathbf{R}^3$ is $\mathbf{R}^3$ itself) so it is an isomorphism.

Summary and Questions

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