14. Lecture 14: 4.3 Coordinate matrix of a linear transformation

Suppose $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis for a vector space (also called a linear space) V. The **$\mathcal{B}$-coordinates of x** are the weights $c_1, \cdots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n, \qquad \text{where} \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

is called the **$\mathcal{B}$-coordinate vector of x**. The coordinate map taking $\mathbf{x}$ is to$[\mathbf{x}]_{\mathcal{B}}$ is denoted

$$L_{\mathcal{B}} : \mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}.$$

Ex Standard basis for the polynomials of degree 2 or less, $\mathbf{P}_2$ is $\{1, t, t^2\}$. We can write $[a + bt + ct^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. All vector space operations in $\mathbf{P}_2$ corresponds to operations in $\mathbf{R}^3$, e.g. adding two polynomials $(-1 + 2t - 3t^2) + (2 + 3t + 5t^2) = 1 + 5t + 2t^2$ corresponds to adding their coordinate vectors: $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$.

Let T be a linear transformation from $V \to V$. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V. Then there is a linear transformation, with matrix B, taking $[\mathbf{x}]_{\mathcal{B}}$ to $[T(\mathbf{x})]_{\mathcal{B}}$:n

$$[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}, \tag{14.1}$$

i.e. if we express $\mathbf{x}$ and $T(\mathbf{x})$ in the basis then the linear transformation of their coefficients;

$$\begin{array}{ccc} \mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n & \xrightarrow{T} & T(\mathbf{x}) = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n \\ \downarrow L_{\mathcal{B}} & & \downarrow L_{\mathcal{B}} \\ [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \end{array}$$

has matrix B. The matrix B is calculated from the above commutative diagram;

$$B = L_{\mathcal{B}} T L_{\mathcal{B}}^{-1}.$$

Substituting $\mathbf{b}_i$ into (14.1) we see that

$$B = \begin{bmatrix} | & | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

Ex Let $T = D$ be the derivative map acting on polynomials of degree ≤ 2, with basis $\mathbf{b}_1 = 1$, $\mathbf{b}_2 = t$ and $\mathbf{b}_3 = t^2$. We have $T(1) = 0$, $T(t) = 1$ and $T(t^2) = 2t$ so $[T(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $[T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $[T(\mathbf{b}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ so $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Change of basis

Ex Suppose we have two basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ that are related by

$$\mathbf{b}_1 = 3\mathbf{c}_1 + 5\mathbf{c}_2, \qquad \mathbf{b}_2 = \mathbf{c}_1 + 2\mathbf{c}_2 \tag{14.2}$$

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, i.e. $\mathbf{x} = \mathbf{b}_1 - 2\mathbf{b}_2$, find $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, such that $\mathbf{x} = d_1\mathbf{c}_1 + d_2\mathbf{c}_2$.

Sol By (14.2) $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and hence

$$[\mathbf{x}]_{\mathcal{C}} = [\mathbf{b}_1 - 2\mathbf{b}_2]_{\mathcal{C}} = [\mathbf{b}_1]_{\mathcal{C}} - 2\,[\mathbf{b}_2]_{\mathcal{C}} = \big[\,[\mathbf{b}_1]_{\mathcal{C}}\;[\mathbf{b}_2]_{\mathcal{C}}\,\big]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The same argument as in the example proves the following:
Th Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two basis for a vector space V. Then there is a unique matrix $S_{\mathcal{B}\to\mathcal{C}}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = S_{\mathcal{B}\to\mathcal{C}}\,[\mathbf{x}]_{\mathcal{B}}$$

The columns of $S_{\mathcal{B}\to\mathcal{C}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$:

$$S_{\mathcal{B}\to\mathcal{C}} = \big[\,[\mathbf{b}_1]_{\mathcal{C}}\;\cdots\;[\mathbf{b}_n]_{\mathcal{C}}\,\big]$$

The matrix $S_{\mathcal{B}\to\mathcal{C}}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.**

If $\mathcal{D}$ is another basis then changing coordinates from $\mathcal{B}$ to $\mathcal{D}$ is the same as changing coordinates first from $\mathcal{B}$ to $\mathcal{C}$ and then from $\mathcal{C}$ to $\mathcal{D}$, so $[\mathbf{x}]_{\mathcal{D}} = S_{\mathcal{C}\to\mathcal{D}}\,[\mathbf{x}]_{\mathcal{C}} = S_{\mathcal{C}\to\mathcal{D}}S_{\mathcal{B}\to\mathcal{C}}\,[\mathbf{x}]_{\mathcal{B}}$ i.e.

$$S_{\mathcal{B}\to\mathcal{D}} = S_{\mathcal{C}\to\mathcal{D}}S_{\mathcal{B}\to\mathcal{C}}.$$

Since $S_{\mathcal{B}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{B}}S_{\mathcal{C}\to\mathcal{B}}$ is the identity transformation the inverse change of variables satisfies

$$S_{\mathcal{B}\to\mathcal{C}} = S^{-1}_{\mathcal{C}\to\mathcal{B}}.$$

If $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis in $\mathbf{R}^n$ and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ another then $[\mathbf{b}_i]_{\mathcal{E}} = \mathbf{b}_i$ so

$$S_{\mathcal{B}\to\mathcal{E}} = \big[\,\mathbf{b}_1 \cdots \mathbf{b}_n\,\big].$$

It follows that

$$S_{\mathcal{B}\to\mathcal{C}} = S_{\mathcal{E}\to\mathcal{C}}S_{\mathcal{B}\to\mathcal{E}} = S^{-1}_{\mathcal{C}\to\mathcal{E}}S_{\mathcal{B}\to\mathcal{E}}$$

In other words we can express a vector in the there different coordinate systems

$$z_1\mathbf{c}_1 + \cdots + z_n\mathbf{c}_n = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$$

and the matrix for the transformation $[\mathbf{x}]_{\mathcal{B}} = (y_1, \dots, y_n) \to [\mathbf{x}]_{\mathcal{C}} = (z_1, \dots, z_n)$ can be obtained by first going $(y_1, \dots, y_n) \to (x_1, \dots, x_n)$ and then $(x_1, \dots, x_n) \to (z_1, \dots, z_n)$. The matrix for the last transformation is easiest obtained as the inverse of the matrix for $(z_1, \dots, z_n) \to (x_1, \dots, x_n)$.

Ex Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be the basis $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Find the change of coordinate matrix from the coordinates in the $\mathcal{B}$ basis to the coordinates in the $\mathcal{C}$ basis.

Sol $S_{\mathcal{B}\to\mathcal{C}} = S^{-1}_{\mathcal{C}\to\mathcal{E}}S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 9 \\ -6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$

Examples of changing coordinates in vector spaces

Ex (a) $T : P_2 \to P_2$ be the linear transformation defined by $T(f) = f + f''$. Let $\mathcal{S} = (1, x, x^2)$ be the standard basis for P_2. Find the $\mathcal{S}$-matrix A for T.

(b) Let $\mathcal{B} = (1 + x, x + x^2, 1 + x^2)$ be another basis for P_2. Let B be the $\mathcal{B}$-matrix for the linear transformation T. Find the invertible matrix S such that $B = S^{-1}AS$.

Sol (a) T maps

$$\begin{array}{ccc} c_1\cdot 1 + c_2\cdot x + c_3\cdot x^2 & \xrightarrow{T} & d_1\cdot 1 + d_2\cdot x + d_3\cdot x^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{\text{multiply by } A} & \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \end{array}$$

Since

$$T(1) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2,$$
$$T(x) = x = 0 \cdot 1 + 1 : x + 0 \cdot x^2,$$
$$T(x^2) = x^2 + 2 = 2 \cdot 1 + 0 \cdot x + 1 \cdot x^2,$$

we have

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{so} \qquad A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Next we want to find the expression for T in the new basis

$$\begin{array}{ccc} a_1\cdot(1 + x) + a_2\cdot(x + x^2) + a_3\cdot(1 + x^2) & \xrightarrow{T} & b_1\cdot(1 + x) + b_2\cdot(x + x^2) + b_3\cdot(1 + x^2) \\ \downarrow & & \downarrow \\ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} & \xrightarrow{\text{multiply by } B} & \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \end{array}$$

Since we already know the matrix for B in the standard coordinates the easiest way to get B is to change coordinates $\quad a_1\cdot(1 + x) + a_2\cdot(x + x^2) + a_3\cdot(1 + x^2) = c_1\cdot 1 + c_2\cdot x + c_3\cdot x^2$

Since the new coordinates already are expressed in terms of the old ones, the easiest way is to get the matrix S from the new coordinates to the old ones $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = S\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

We get $\quad S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad S\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{so} \qquad S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Multiplication B corresponds to $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \xrightarrow{\text{multiply by } S} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \xrightarrow{\text{multiply by } S^{-1}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

so $B = S^{-1}AS = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \ldots.$ For more info see Ex 5, Ex 8 in sec 4.3.

Also look at Ex 7 in sec 4.3 about changing basis in a plane.

Summary and Questions

Let T be a linear transformation from $V \to V$. Suppose $\mathcal{B}=\{\mathbf{b}_1,\dots,\mathbf{b}_n\}$ is a basis for V. Then there is a linear transformation, with matrix B, taking $[\mathbf{x}]_{\mathcal{B}}$ to $[T(\mathbf{x})]_{\mathcal{B}}$

$$[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}\,,$$

i.e. if we express $\mathbf{x}$ and $T(\mathbf{x})$ in the basis then the linear transformation of their coefficients;

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\downarrow L_{\mathcal{B}} & & \downarrow L_{\mathcal{B}} \\
[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}
\end{array}$$

has matrix B, i.e.

$$B = L_{\mathcal{B}} T L_{\mathcal{B}}^{-1}.$$

Substituting $\mathbf{b}_i$ into (14.1) we see that

$$B = \begin{bmatrix} | & | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

Ex The map $T = D$ the derivative acting on polynomials of degree ≤ 2 with the basis $\mathbf{b}_1 = 1$, $\mathbf{b}_2 = x$ and $\mathbf{b}_3 = x^2$. then $T(a+bx+cx^2) = b+2cx$ so the B matrix is $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

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Sol

$$[\mathbf{x}]_{\mathcal{C}} = [\mathbf{b}_1 - 2\mathbf{b}_2]_{\mathcal{C}} = [\mathbf{b}_1]_{\mathcal{C}} - 2\,[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

since $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.