19. Lecture 19: 5.5: Inner products and Fourier series

Def An **inner product** on a vector space V is a function that for each pair of vectors gives a real number: $V \ni \mathbf{f}, \mathbf{g} \to \langle \mathbf{f}, \mathbf{g} \rangle \in \mathbf{R}$, satisfying:
(i) $\langle \mathbf{f}, \mathbf{f} \rangle \geq 0$ with equality if and only if $\mathbf{f} = 0$.
(ii) $\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle$.
(iii) $\langle \alpha\mathbf{f} + \beta\mathbf{g}, \mathbf{h} \rangle = \alpha\langle \mathbf{f}, \mathbf{h} \rangle + \beta\langle \mathbf{g}, \mathbf{h} \rangle$.
An **inner product space** is a vector space with an inner product.

Ex 1 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$.
Ex 2 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 w_1 + \cdots + x_n y_n w_n$, where $w_i > 0$, for $i = 1, ..., n$.

Recall that the book uses the names linear space and elements for what traditionally is called vector space and vectors. A function is not a vector in a traditional sense but thinking of it as a vector one can use the methods developed for finite dimensional vector spaces.
One can think a continuous function $f \in C[a, b]$ as a vector with infinitely many components $f(t)$, for every $t \in [a, b]$. One can think of a continuous function as approximated by its values $\{f(t_0), f(t_1), \dots, f(t_n)\}$ at finitely many points $t_0 < t_1 < \cdots < t_n$, (e.g. $t_k = a + k\triangle t$ and $\triangle t = (b-a)/n$), or by polynomials of degree $\leq n$ with the same values at these points.

Ex 3 $f, g \in C[a, b]$, the continuous functions on the interval $[a, b]$, and $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$.
(ii) and (iii) in the definition are easy to check. (i) follows from using the fundamental theorem of calculus. However it can also be seen from the definition of the Riemann integral as a limit of Riemann sums

$$\int_a^b f(t)g(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^n f(t_k)g(t_k)\,\triangle t = \lim_{n\to\infty} \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_n) \end{bmatrix} \triangle t$$

Def $\mathbf{f}$ and $\mathbf{g}$ are called **orthogonal** if $\langle \mathbf{f}, \mathbf{g} \rangle = 0$.
Def The **norm** is defined to be $\|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$.
Def The **distance** between $\mathbf{f}$ and $\mathbf{g}$ is defined to be $\mathrm{dist}(\mathbf{f}, \mathbf{g}) = \|\mathbf{f} - \mathbf{g}\|$.

Note that **Phytagorean theorem** holds: $\|\mathbf{f} + \mathbf{g}\|^2 = \|\mathbf{f}\|^2 + \|\mathbf{g}\|^2$, if $\langle \mathbf{f}, \mathbf{g} \rangle = 0$.
This follows since $\|\mathbf{f} + \mathbf{g}\|^2 = \langle \mathbf{f} + \mathbf{g}, \mathbf{f} + \mathbf{g} \rangle = \langle \mathbf{f}, \mathbf{f} \rangle + \langle \mathbf{g}, \mathbf{g} \rangle + \langle \mathbf{f}, \mathbf{g} \rangle + \langle \mathbf{g}, \mathbf{f} \rangle = \|\mathbf{f}\|^2 + \|\mathbf{g}\|^2$.
Cauchy-Schwarz inequality also holds: $|\langle \mathbf{f}, \mathbf{g} \rangle| \leq \|\mathbf{f}\|\,\|\mathbf{g}\|$.
This follows from the Phytagorean theorem by writing $\mathbf{f} = \langle \mathbf{f}, \mathbf{g} \rangle \langle \mathbf{g}, \mathbf{g} \rangle^{-1} \mathbf{g} + \mathbf{h}$, where $\langle \mathbf{g}, \mathbf{h} \rangle = 0$.

Question Given $f \in C[a, b]$ which is the polynomial $p \in P_n$ of degree $\leq n$ closest to f, i.e. so $\mathrm{dist}(f, p) = \|\mathbf{f} - \mathbf{p}\|$ is as small as possible, where the norm is from the inner product in Ex 3?

The orthogonal decomposition theorem Let W be a subspace of a vector space V and let $\{\mathbf{g}_1, \dots, \mathbf{g}_p\}$ be an orthnormal basis for W. Any $\mathbf{f} \in V$ can be written uniquely as

$$\mathbf{f} = \mathbf{g} + \mathbf{h}, \qquad \mathbf{g} \in W, \quad \mathbf{h} \in W^\perp$$

where

$$\mathbf{g} = \mathrm{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{g}_1 \rangle \mathbf{g}_1 + \cdots + \langle \mathbf{f}, \mathbf{g}_p \rangle \mathbf{g}_p, \quad \text{and} \quad W^\perp = \{\mathbf{h} \in V;\ \langle \mathbf{h}, \mathbf{g}_1 \rangle = 0, \dots, \langle \mathbf{h}, \mathbf{g}_p \rangle = 0\}$$

are the orthogonal projection of $\mathbf{f}$ onto W respectively the orthogonal complement of W.

The best approximation theorem Let W be a subspace of a vector space V and $\mathbf{f} \in V$. Then the orthogonal projection $\mathrm{proj}_W \mathbf{f}$ of $\mathbf{f}$ onto W is the vector in W closest to $\mathbf{f}$, i.e.

$$\|\mathbf{f} - \mathrm{proj}_W \mathbf{f}\| < \|\mathbf{f} - \mathbf{g}\|, \qquad \text{for all} \quad \mathbf{g} \in W, \quad \mathbf{g} \neq \mathrm{proj}_W \mathbf{f}.$$

Ex 4 Find the polynomial P_2 of degree ≤ 2 that best approximates the polynomial $f(t) = t^4$ on the interval $[-1, 1]$ in the inner product in Ex 3.
Sol An orthonormal basis for P_2 in the inner product in Ex 3 can be found using Gram-Schmidt to be $p_0 = 1/\sqrt{2}$, $p_1 = \sqrt{3/2}\, t$, $p_2 = (3t^2 - 1)\sqrt{5/8}$. Then $\langle f, p_0\rangle = \sqrt{2}/5$, $\langle f, p_1\rangle = 0$ and $\langle f, p_2\rangle = \big(3/7 - 1/5\big)\sqrt{5/2}$. Hence
$$\mathrm{proj}_{P_2}(f)(t) = \langle f, p_0\rangle p_0(t) + \langle f, p_1\rangle p_1(t) + \langle f, p_2\rangle p_2(t) = \frac{1}{5} + \big(\frac{3}{7} - \frac{1}{5}\big)\frac{5}{4}(3t^2 - 1) = \frac{6}{7}t^2 - \frac{3}{35}.$$

Fourier Series

Let T_n be the subspace of $V = C[-\pi, \pi]$ spanned by all trigonometric polynomials up to order n: $1, \cos t, \dots, \cos nt, \sin t, \dots, \sin nt$, i.e. T_n consists of all functions of the form
$$\frac{a_0}{2} + a_1 \cos t + \cdots + a_n \cos nt + b_1 \sin t + \cdots + b_n \sin nt$$
The basis vectors $1, \cos t, \dots, \cos nt, \sin t, \dots, \sin nt$, are orthogonal to each other, i.e.
$$\int_{-\pi}^{\pi} \cos kt \sin \ell t\, dt = 0$$
$$\int_{-\pi}^{\pi} \cos kt \cos \ell t\, dt = \begin{cases} \pi, & \text{if } k = \ell, \\ 0, & \text{if } k \neq \ell, \end{cases}$$
$$\int_{-\pi}^{\pi} \sin kt \sin \ell t\, dt = \begin{cases} \pi, & \text{if } k = \ell, \\ 0, & \text{if } k \neq \ell, \end{cases}$$
Using Euler's formulas, $\cos kt = \dfrac{e^{ikt} + e^{-ikt}}{2}$, $\sin kt = \dfrac{e^{ikt} - e^{-ikt}}{2i}$, the proof reduces to
$$\int_{-\pi}^{\pi} e^{i(k\pm\ell)t}\, dt = \begin{cases} 2\pi, & \text{if } k \pm \ell = 0, \\ 0, & \text{if } k \pm \ell \neq 0, \end{cases}$$
The orthogonal projection of f onto T_n is given by
$$\mathrm{proj}_{T_n}(\mathbf{f}) = \frac{\langle \mathbf{f}, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\langle \mathbf{f}, \cos t\rangle}{\langle \cos t, \cos t\rangle}\cos t + \cdots + \frac{\langle \mathbf{f}, \cos nt\rangle}{\langle \cos nt, \cos nt\rangle}\cos nt$$
$$+ \frac{\langle \mathbf{f}, \sin t\rangle}{\langle \sin t, \sin t\rangle}\sin t + \cdots + \frac{\langle \mathbf{f}, \sin nt\rangle}{\langle \sin nt, \sin nt\rangle}\sin nt$$

Ex 5 Expand the step function $f(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ -1, & \text{if } t < 0 \end{cases}$ in a Fourier series on $[-\pi, \pi]$.($\notin C[-\pi, \pi]$)

Since $f(t)$ is an odd function and $\cos kt$ is an even function it follows that $\langle f(t), \cos kt\rangle = 0$. Moreover since $\sin kt$ is an odd function we have
$$\langle f(t), \sin kt\rangle = \int_{-\pi}^{\pi} f(t) \sin kt\, dt = 2\int_0^{\pi} \sin kt\, dt = -\frac{2}{k}\cos kt\Big|_0^{\pi} = -\frac{2}{k}\big((-1)^k - 1\big)$$
Hence the orthogonal projection of f onto T_n, where n is odd is given by
$$\mathrm{proj}_{T_n}(\mathbf{f}) = \frac{\langle \mathbf{f}, \sin t\rangle}{\langle \sin t, \sin t\rangle}\sin t + \cdots + \frac{\langle \mathbf{f}, \sin nt\rangle}{\langle \sin nt, \sin nt\rangle}\sin nt = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \cdots + \frac{4}{n\pi}\sin nt$$

Summary

Def An **inner product** on a vector space V is a function that for each pair of vectors gives a real number: $V \ni \mathbf{f}, \mathbf{g} \to \langle \mathbf{f}, \mathbf{g} \rangle \in \mathbf{R}$, satisfying:
(i) $\langle \mathbf{f}, \mathbf{f} \rangle > 0$ if $\mathbf{f} \neq 0$, (ii) $\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle$, (iii) $\langle \alpha \mathbf{f} + \beta \mathbf{g}, \mathbf{h} \rangle = \alpha \langle \mathbf{f}, \mathbf{h} \rangle + \beta \langle \mathbf{g}, \mathbf{h} \rangle$.

Ex 1 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$.

One can think a continuous function $f \in C[a,b]$ as a vector with infinitely many components $f(t)$, for every $t \in [a,b]$. One can think of a continuous function as approximated by its values $\{f(t_0), f(t_1), \ldots, f(t_n)\}$ at finitely many points $t_0 < t_1 < \cdots < t_n$, (e.g. $t_k = a + k\triangle t$ and $\triangle t = (b-a)/n$), or by polynomials of degree $\leq n$ with the same values at these points.

Ex 3 $f, g \in C[a,b]$, the continuous functions on the interval $[a,b]$, and $\langle f, g \rangle = \displaystyle\int_a^b f(t) g(t)\, dt$.

The Riemann integral is a limit of Riemann sums

$$\int_a^b f(t)g(t)\,dt = \lim_{n\to\infty} \sum\nolimits_{k=1}^n f(t_k) g(t_k) \triangle t = \lim_{n\to\infty} \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_n) \end{bmatrix} \triangle t$$

Def $\mathbf{f}$ and $\mathbf{g}$ are called **orthogonal** if $\langle \mathbf{f}, \mathbf{g} \rangle = 0$.
Def The **norm** is defined to be $\|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$.
Def The **distance** between $\mathbf{f}$ and $\mathbf{g}$ is defined to be $\text{dist}(\mathbf{f}, \mathbf{g}) = \|\mathbf{f} - \mathbf{g}\|$.

Question Given $f \in C[a,b]$ which is the polynomial $p \in P_n$ of degree $\leq n$ closest to f, i.e. so $\text{dist}(f, p) = \|\mathbf{f} - \mathbf{p}\|$ is as small as possible, where the norm is from the inner product in Ex 3?

The orthogonal decomposition theorem Let W be a subspace of a vector space V and let $\{\mathbf{g}_1, \ldots, \mathbf{g}_p\}$ be an orthnormal basis for W. Any $\mathbf{f} \in V$ can be written uniquely as

$$\mathbf{f} = \mathbf{g} + \mathbf{h}, \qquad \mathbf{g} \in W, \quad \mathbf{h} \in W^\perp$$

where

$$\mathbf{g} = \text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{g}_1 \rangle \mathbf{g}_1 + \cdots + \langle \mathbf{f}, \mathbf{g}_p \rangle \mathbf{g}_p, \quad \text{and} \quad W^\perp = \{\mathbf{h} \in V;\ \langle \mathbf{h}, \mathbf{g}_1 \rangle = 0, \ldots, \langle \mathbf{h}, \mathbf{g}_p \rangle = 0\}$$

are the orthogonal projection of $\mathbf{f}$ onto W respectively the orthogonal complement of W.

The best approximation theorem Let W be a subspace of a vector space V and $\mathbf{f} \in V$. Then the orthogonal projection $\text{proj}_W \mathbf{f}$ of $\mathbf{f}$ onto W is the vector in W closest to $\mathbf{f}$, i.e.

$$\|\mathbf{f} - \text{proj}_W \mathbf{f}\| < \|\mathbf{f} - \mathbf{g}\|, \qquad \text{for all} \quad \mathbf{g} \in W, \quad \mathbf{g} \neq \text{proj}_W \mathbf{f}.$$

Fourier Series

Let T_n be the subspace of $C[-\pi, \pi]$ spanned by all trigonometric polynomials of order $\leq n$:

$$p_n(t) = a_0/2 + a_1 \cos t + \cdots + a_n \cos nt + b_1 \sin t + \cdots + b_n \sin nt$$

The basis vectors $1, \cos t, \ldots, \cos nt, \sin t, \ldots, \sin nt$, are orthonormal in $\langle f, g \rangle = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(t) g(t)\, dt$.
If $f \in C[-\pi, \pi]$ then with $a_k = \langle f, \cos kt \rangle$ and $b_k = \langle f, \sin kt \rangle$ we have

$$\|f - p_n\| < \|f - q\|, \qquad \text{for all} \quad q \in T_n, \quad q \neq p_n.$$

Question When is $f(t) = \lim_{n\to\infty} p_n(t)$?
Answer If $f \in C^1[-\pi, \pi]$ (i.e. has a continuous derivative) then the sum converges pointwise. If $f \in L^2$ i.e. $\|f\| < \infty$ then the sum converges in L^2, i.e. $\lim_{n\to\infty} \|f - p_n\| = 0$.
The Fourier series for the step function in Ex 5 does not converge pointwise at the origin because the step function is not continuous there but it converges in the integrated norm. Proving these things requires a course in Analysis (advanced Calculus).