2. Lecture 2: 1.2 Gauss-Jordan elimination, Row-Echelon form

Last time we saw that we could reduce an $n \times n$ system by row operations into an equivalent system in triangular form, e.g.

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

If the diagonal entries are non-zero this can be solved by back-substitution.
However, this fails if we get a diagonal entry that is 0, e.g.

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 0 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

There are no solutions since the third equation is a contradiction.
Moreover, $m \times n$ systems, with $m \neq n$, can not be written in triangular form, e.g.

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

In this case there are infinitely many solutions, in fact x_2 can be chosen freely.
In either case the systems are in a form so its easy to determine the solution set.

A matrix is said to be in **Row Echelon Form** ("step-like form") if
Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
A triangular matrix is in row echelon form and so is:

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is easy to determine if a system with augmented matrix in row-echelon form is consistent, and if so to solve it by back-substitution.

A matrix is said to be in **Reduced Row Echelon Form** if
(i) It is in row echelon form, and (ii) Each leading non-zero entry is 1 and
(iii) The leading entry in each row is the only non-zero entry in its column.
A diagonal matrix with 1's in the diagonal is in row echelon form and so is:

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If the matrix is in reduced row echelon form we do not need to do back-substitution.

Recall the **Elementary Row Operations** on a matrix are
1. Add to one row a multiple of another.
2. Interchange two rows
3. Multiply all entries in a row by the same nonzero number.
Two matrices are said to be **row equivalent** if one can be transformed into the other by elementary row operations.
Th If the augmented matrices for two systems are row equivalent then they have the same solution set, i.e. elementary row operations don't change the solution set.
Th Each matrix is row-equivalent to a unique matrix in reduced row echelon form.
Note however that the row echelon form is not unique.

The process of using row operations to transform a matrix to (reduced) row echelon form is generally known as Gaussian elimination, although it turned out the Chinese were using this method 2000 years earlier.

Gauss is one of the greatest mathematicians of all times with fundamental contributions to number theory, astronomy and geometry. He lived in Germany 1777-1855. He is said to have been able to do arithmetic before he could speak. At 3 he corrected a mistake in the payroll for his fathers company but his father didn't think much of his genius.

How Gauss developed his elimination method is noteworthy. An astronomer Piazzi discovered what he believed was a new planet and was able to observe its path for only 40 days. From these limited observations Gauss was able to predict were the astroid would return a year later. In the course of his computations Gauss had to solve a system of 17 linear equations. In dealing with this problem he also used the method of least square approximation that he previously developed. We will learn this method later in the course. Since Gauss at first refused to reveal his method some people accused him of sorcery.

A **pivot position** in a matrix is a place corresponding to a leading 1 in the reduced row echelon form. A **pivot column** is a column that contains a pivot position.
A **pivot** is a nonzero number in a pivot position (it is also the first nonzero number in the row that does the elimination).
The **basic variables** or **leading variables** are the variables corresponding to the pivot columns and the **free variables** are the other variables.

$$\begin{aligned} x_1 \qquad - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad\qquad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_1 and x_2 are lead variables and x_3 is a free variable, that can be chosen freely:

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is a free parameter} \end{cases}$$

Th A linear system is consistent (i.e. there is one or more solutions) if and only if the echelon form of the augmented matrix has no row of the form

$$\left[\begin{array}{ccc|c} 0 & \dots & 0 & b \end{array}\right] \quad b \neq 0$$

If consistent there is one solution if no free variables and infinitely many if free variables.

Ex Solve the system

$$\begin{aligned} x_3 - \quad x_4 - \quad x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 &= 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 &= 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 &= 6 \end{aligned} \qquad \left[\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right]$$

Sol. Interchange row one and two so we have a nonzero pivot

$$\left[\begin{array}{ccccc|c} 2 & 4 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \begin{array}{l} (2) \\ (1) \\ \\ \\ \end{array}$$

Divide row one by 2

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \begin{array}{l} (1)/2 \\ \\ \\ \\ \end{array}$$

Eliminate all other entries in the pivot column

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \\ (3)-2(1) \\ (4)-3(1) \end{array}$$

Since there only zeros in the second column below the first row the pivot column is now the third column and we use the pivot element on the second row in the third column to eliminate the other entries in the pivot column

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{array}\right] \begin{array}{l} (1)-(2) \\ \\ (3)-(2) \\ (4)-3(2) \end{array}$$

Next we divide row three by to to get one as pivot entry

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{array}\right] \begin{array}{l} \\ \\ (3)/2 \\ \\ \end{array}$$

and eliminate the other entries from the pivot column

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (1)-2(3) \\ (2)+(3) \\ \\ (4)-6(3) \end{array}$$

The augmented matrix is now in reduced row-echelon form and the system is

$$\begin{aligned} x_1 + 2x_2 \qquad + 3x_4 \qquad &= 2 \\ x_3 - \ x_4 \qquad &= 2 \\ x_5 &= -2 \end{aligned}$$

Here x_1, x_3, x_5 are lead variables and x_2, x_3 are free variables

$$\begin{aligned} x_1 &= 2 - 2x_2 - 3x_4 \\ x_3 &= 2 + x_4 \\ x_5 &= -2 \end{aligned}$$

1.3: Rank

The **rank** of a matrix is the number of leading 1's in the reduced row-echelon form.

If A is an $m \times n$ matrix then it is clear that $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$. Moreover, since

$$\text{number of lead variables} + \text{number of free variables} = \text{number of columns},$$

we have

$$\text{number of free variables} = n - \text{rank}(A).$$

If $m = n$, i.e. m equations in n unknowns, and $\text{rank}(A) = n$ then we have exactly one solution.

Summary and Conceptual Questions

A matrix is said to be in Row Echelon Form ("step-like form") if it looks like

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A matrix is said to be in Reduced Row Echelon Form if it looks like

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Any matrix can be transformed using the elementary row operations into a unique matrix in reduced row echelon form (*rref*).

A linear system is consistent (i.e. there is one or more solutions) if and only if the row echelon form of the augmented matrix has no row of the form

$$\left[\begin{array}{ccc|c} 0 & \dots & 0 & b \end{array}\right] \quad b \neq 0$$

Consider the system in reduced row echelon form

$$\begin{aligned} x_1 + 2x_2 \qquad + 3x_4 \qquad &= 2 \\ x_3 - \ x_4 \qquad &= 2 \\ x_5 &= -2 \end{aligned}$$

Here x_1, x_3, x_5 are called lead variables (leading in their row) and x_2, x_3 are called free variables (free to choose)

$$\begin{aligned} x_1 &= 2 - 2x_2 - 3x_4 \\ x_3 &= 2 + x_4 \\ x_5 &= -2 \end{aligned}$$

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The **rank** of a matrix is the number of leading 1's in the reduced row-echelon form.
If A is an $m \times n$ matrix then it is clear that $\operatorname{rank}(A) \leq m$ and $\operatorname{rank}(A) \leq n$. Moreover, since

$$\text{number of lead variables} + \text{number of free variables} = \text{number of columns},$$

we have

$$\text{number of free variables} = n - \operatorname{rank}(A).$$

If $m = n$, i.e. m equations in n unknowns, and $\operatorname{rank}(A) = n$ then we have exactly one solution.

Question: If $m > n$ and $\operatorname{rank}(A) = n$ which statement of the following statements is correct: (1) at most one solution, or (2) at least one solution.
Question: If $m = n$ and $\operatorname{rank}(A) < n$ which of the following situations can occur (more than one can be correct) (1) exactly one solution, (2) infinitely many solutions, or (3) no solutions.