24. Lecture 24 7.2-3 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A vector $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvector** and λ an **eigenvalue** if

$$A\mathbf{x} = \lambda \mathbf{x},$$

i.e. multiplication with A acts in a very simple way on $\mathbf{x}$. The goal is to find n linearly independent eigenvectors

$$A\mathbf{x}_k = \lambda_k \mathbf{x}_k, \qquad k = 1, \dots, n.$$

If $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ form a basis then any vector $\mathbf{x}$ can be expressed in term of them and so the matrix A is completely determined by its action on the n eigenvectors:

$$A\big(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n\big) = c_1 A\mathbf{x}_1 + \cdots + c_n A\mathbf{x}_n = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n.$$

A scalar λ is an eigenvalue if and only if there is $\mathbf{x} \neq \mathbf{0}$ such that

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The set of all solutions $E_\lambda = \mathrm{Ker}(A - \lambda I)$ is called the **eigenspace** corresponding to eigenvalue λ. Existence of a nontrivial solution is equivalent to that $A - \lambda I$ is not invertible which is equivalent to

$$p_A(\lambda) \equiv \det(A - \lambda I) = 0.$$

The **characteristic polynomial** $p_A(\lambda)$ for the matrix A, is a polynomial of degree n, and its roots are exactly the eigenvalues of A.

In the case of 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc)$$

The sum of the diagonal elements of a square matrix $A = (a_{ij})$ is called the trace $\mathrm{Tr}\, A = a_{11} + \dots a_{nn}$. In case of a 2×2 matrix we have

$$p_A(\lambda) = \det(A - \lambda I) = \lambda^2 - (\mathrm{Tr}\, A)\lambda + \det A$$

In the case of $n \times n$

$$p_A(\lambda) = \det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n-1}(\mathrm{Tr}\, A)\lambda^{n-1} + \cdots + \det A$$

Th If $p_A(\lambda)$ has n different real roots then A has n different linearly independent eigenvectors.

Ex 1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Sol The eigenvalues are solution of the characteristic equation:

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2^2 = (1-\lambda-2)(1-\lambda+2) = 0$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 3$. The eigenvectors are solutions to $(A - \lambda_i)\mathbf{x}_i = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{c} 2x_1 - 2x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{array} \quad \Leftrightarrow \quad \mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{c} -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_2 = 0 \end{array} \quad \Leftrightarrow \quad \mathbf{x}_2 = \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Ex 2 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Sol This is the matrix for a rotation $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with an angle $\theta = \pi/4$ and can not have any real eigenvectors unless the rotation a multiple of π.

The eigenvalues are solution of:

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1^2 = (\lambda - i)(\lambda + i) = 0$, i.e. $\lambda_1 = i$, or $\lambda_2 = -i$.

Ex 3 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Sol The eigenvalues are solution of the characteristic equation:

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$

The eigenvalues are $\lambda_1 = \lambda_2 = 0$. The eigenvectors are solutions to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad y = 0$$

so the only linearly independent eigenvector is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Ex 4 Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Sol Expanding along the first row

$$p_A(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\big((3-\lambda)^2 - 1\big)$$

Characteristic polynomial $(2-\lambda)^2(4-\lambda)$.

$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and $(A-2I)\mathbf{x} = \mathbf{0}$ has augmented matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow x_1 - x_2 - x_3 = 0$ and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ span the eigenspace.

$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, augmented matrix $\begin{bmatrix} -2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow$ $\begin{cases} x_1 = 0, \\ x_2 - x_3 = 0 \end{cases}$ and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Def We say that an eigenvalue λ_0 has **algebraic multiplicity** k if λ_0 is a root of multiplicity k of the characteristic polynomial $p_A(\lambda)$, meaning that we can write

$$p_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda),$$

for some polynomial $g(\lambda)$ with $g(\lambda_0) \neq 0$. We write $\text{almu}(\lambda_0) = k$.

Def Consider an eigenvalue λ. Then the kernel of $A - \lambda I$ is called the **Eigenspace** of A and denoted

$$E_\lambda = \text{Ker}(A - \lambda I)$$

Def The **geometric multiplicity** of an eigenvalue is the dimension of the corresponding Eigenspace.

Th The geometric multiplicity is always less than or equal to the algebraic multiplicity.

Summary

Let A be an $n \times n$ matrix. A vector $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvector** and λ an **eigenvalue** if

$$A\mathbf{x} = \lambda\mathbf{x},$$

i.e. multiplication with A acts in a very simple way on $\mathbf{x}$. The goal is to find n linearly independent eigenvectors

$$A\mathbf{x}_k = \lambda_k \mathbf{x}_k, \qquad k = 1, \dots, n.$$

If $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ form a basis then any vector $\mathbf{x}$ can be expressed in term of them and so the matrix A is completely determined by its action on the n eigenvectors:

$$A\big(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n\big) = c_1 A\mathbf{x}_1 + \cdots + c_n A\mathbf{x}_n = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n.$$

A scalar λ is an eigenvalue if and only if there is $\mathbf{x} \neq \mathbf{0}$ such that

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The set of all solutions $E_\lambda = \mathrm{Ker}(A - \lambda I)$ is called the **eigenspace** corresponding to eigenvalue λ. Existence of a nontrivial solution is equivalent to that $A - \lambda I$ is not invertible which is equivalent to

$$p_A(\lambda) \equiv \det(A - \lambda I) = 0.$$

The **characteristic polynomial** $p_A(\lambda)$ for the matrix A, is a polynomial of degree n, and its roots are exactly the eigenvalues of A.

Th If $p_A(\lambda)$ has n different real roots then A has n different linearly independent eigenvectors.

Ex 1 If $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ then $p_A(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1+\lambda)(\lambda - 3)$, so $\lambda_1 = -1$, $\lambda_2 = 3$.

The eigenvectors are solutions to $(A - \lambda_i)\mathbf{x}_i = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Leftrightarrow\quad \begin{matrix} 2x_1 - 2x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{matrix} \quad\Leftrightarrow\quad \mathbf{x}_1 = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Leftrightarrow\quad \begin{matrix} -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_2 = 0 \end{matrix} \quad\Leftrightarrow\quad \mathbf{x}_2 = \beta\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Ex 2 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ corresponds to a rotation $\pi/2$ so it has no eigenvectors and $p_A(\lambda) = \lambda^2 + 1$.

Ex 3 If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $p_A(\lambda) = \lambda^2$. It has only one linearly independent eigenvector

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Leftrightarrow\quad x_2 = 0 \quad\Leftrightarrow\quad \mathbf{x} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Ex 4 If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $p_A(\lambda) = (1-\lambda)^2$ and every vector is an eigenvector with eigenvalue 1.

Def An eigenvalue λ_0 has **algebraic multiplicity** k for some polynomial g if

$$p_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda), \qquad \text{where} \quad g(\lambda_0) \neq 0.$$

Def The **geometric multiplicity** of an eigenvalue λ is the dimension of its **Eigenspace**:

$$E_\lambda = \mathrm{Ker}(A - \lambda I)$$

Th The geometric multiplicity is always less than or equal to the algebraic multiplicity.