29. Lecture 29: 8.2 Quadratic Forms

Min-Max

A function $F(x, y)$ has a **stationary or critical point** at (x_0, y_0) if the first order derivatives vanishes $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$. The question we want to ask is if the critical point is a local **maximum** or **minimum** or neither. For simplicity of notation we will assume that the critical point is $(0, 0)$. By Taylor's formula

$$F(x, y) = F(0, 0) + F_x(0, 0)x + F_y(0, 0)y + P_2(x, y) + R_2(x, y),$$

where

$$P_2(x, y) = \frac{1}{2}\big(F_{xx}(0, 0)x^2 + 2F_{xy}(0, 0)xy + F_{yy}(0, 0)y^2\big),$$

and the error is smaller when (x, y) is close to $(0, 0)$.

$$|R_2(x, y)| \le C(|x| + |y|)^3.$$

The behavior of F close to $(0, 0)$ is determined by the behavior of P_2.

Therefore we want to study for which constants a, b, c a polynomial

$$q(x, y) = ax^2 + 2bxy + cy^2,$$

has a max/min or saddle point.
Ex The typical examples are $q = x^2 + y^2$ a max, $q = -(x^2 + y^2)$ a min and $q(x, y) = \pm(x^2 - y^2)$ or $q = 2xy$ a saddle point.

The **quadratic form** $q(x, y)$ is said to be **positive definite** if $q(x, y) > 0$ for $(x, y) \neq (0, 0)$.
What conditions on a, b, c are needed for $q(x, y)$ to be positive definite?
We must have $q(1, 0) = a > 0$ and $q(0, 1) = c > 0$ but that is not all.
If we complete the square we get

$$q(x, y) = a\Big(x + \frac{b}{a}y\Big)^2 + \Big(c - \frac{b^2}{a}\Big)y^2,$$

so we must have $q\big(-b/a, 1\big) = c - b^2/a > 0$ so we must also have that $ac > b^2$.
On the other hand it follows that $q(x, y)$ is positive definite if these conditions are true.

Now $q(x, y)$ can be written as a bilinear form using a symmetric matrix:

$$q(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}, \qquad \text{if} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

so the condition for positivity is that the determinant $ac - b^2 > 0$ and $a > 0$. An equivalent condition is that the eigenvalues λ_1 and λ_2 are positive since $\lambda_1\lambda_2 = ac - b^2$ and $\lambda_1 + \lambda_2 = a + c$.

Quadratic Forms and Positive definiteness

If A is a symmetric $n \times n$ matrix then the **quadratic form**

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i,j=1}^n a_{ij} x_i x_j,$$

is said to be **positive definite** if $q(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$.

Th That $q(\mathbf{x})$ is positive definite equivalent to that all the eigenvalues of A $\lambda_i > 0$.
Pf In fact A can be diagonalized $A = QDQ^T$, $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ are the eigenvalues so if we set $\mathbf{y} = Q^T\mathbf{x}$ we obtain

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T QDQ^T\mathbf{x} = (Q^T\mathbf{x})^T DQ\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = \widetilde{q}(\mathbf{y})$$

which is always positive if the eigenvalues are positive and $\mathbf{y} \neq 0$, which is equivalent to $\mathbf{x} \neq 0$.

Ex Is $q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + 4x_1x_2 + 4x_2x_3$ positive definite?
Sol Because of all the plus signs this form looks positive definite, but the matrix of the form

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix},$$

has the eigenvalues 5, 2, and -1, so q is an indefinite quadratic form, not positive definite.

Rem Like in the two dimensional case there is an easier way to check if it is positive definite:

Th Principal submatrices and definiteness Consider a symmetric $n \times n$ matrix A. For $m = 1, ..., n$, let $A^{(m)}$ be the $m \times m$ matrix obtained by omitting all rows and columns of A past the mth. These matrices $A^{(m)}$ are called the principal submatrices of A. The matrix A is positive definite if (and only if) $\det(A^{(m)}) > 0$, for all $m = 1, ..., n$.
Pf Problem 34 in the book.

Ex In the example $\det A^{(1)} = 3 > 0$, $\det A^{(2)} = 3{\cdot}2 - 2{\cdot}2 = 2 > 0$ and $\det A^{(3)} = \det A = 6 > 0$.

Constrained optimization

Let $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is symmetric, be a quadratic form. Find the maximum and minimum of $q(\mathbf{x})$ when $\|\mathbf{x}\| = 1$.

Ex Find the max and min of $q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ when $x_1^2 + x_2^2 + x_3^2 = 1$.
Sol We have $q(\mathbf{x}) = 9x_1^2+4x_2^2+3x_3^2 \leq 9(x_1^2+x_2^2+x_3^2) = 9$, when $x_1^2+x_2^2+x_3^2 = 1$, so max≤ 9.
On the other hand if $(x_1, x_2, x_3) = (1, 0, 0)$ then $q(\mathbf{x}) = 9$ and $x_1^2+ x_2^2+x_3^2 = 1$, so max$= 9$.
Similarly $q(\mathbf{x}) = 9x_1^2+ 4x_2^2+ 3x_3^2 \geq 3(x_1^2+ x_2^2+ x_3^2) = 3$, when $x_1^2+ x_2^2+ x_3^2 = 1$, so min ≥ 3.
On the other hand if $(x_1, x_2, x_3) = (0, 0, 1)$ then $q(\mathbf{x}) = 3$ and $x_1^2+ x_2^2+ x_3^2 = 1$, so min $= 3$.

Rem Note that the max and min of $q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ when $x_1^2 + x_2^2 + x_3^2 = 1$ is the largest respectively smallest eigenvalue of $A = \text{diag}\{9, 4, 3\}$. This is always true:

Th Let A be a symmetric matrix, and define m and M by

$$m = \min\{\mathbf{x}^T A\mathbf{x};\ \|\mathbf{x}\| = 1\}, \qquad M = \max\{\mathbf{x}^T A\mathbf{x};\ \|\mathbf{x}\| = 1\}.$$

Then M is the greatest eigenvalue of A and m is the least eigenvalue of A.
The value of $\mathbf{x}^T A\mathbf{x}$ is M when $\mathbf{x}$ is a unit eigenvector corresponding to M.
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Pf Orthogonally diagonalize A as QDQ^T. We know that

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y}, \qquad \text{when } \mathbf{x} = Q\mathbf{y}$$

Also,

$$\|\mathbf{x}\| = \|Q\mathbf{y}\| = \|\mathbf{y}\|$$

because $Q^T Q = I$. In particular $\|\mathbf{x}\| = 1$ if and only if $\|\mathbf{y}\| = 1$. Thus $\mathbf{x}^T A\mathbf{x}$ and $\mathbf{y}^T D\mathbf{y}$ assume the same values as $\mathbf{x}$ and $\mathbf{y}$ range over all unit vectors. We have therefore reduced proving the theorem for diagonal matrices where the result follows as in the previous example.

Summary

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