3. Lecture 3: (1.2) 1.3 Vector Equations and Matrix multiplication

In linear algebra we think of **vectors** in $\mathbf{R}^n$ as column vectors or $n \times 1$ matrices

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Addition and **scalar multiplication** are defined by

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \qquad \lambda\mathbf{u} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \vdots \\ \lambda u_n \end{bmatrix}, \quad \lambda \in \mathbf{R}.$$

Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ and scalars $\lambda_1, \dots, \lambda_k$, the vector

$$\mathbf{w} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, with weights $\lambda_1, \dots \lambda_k$.

The first question we will ask today is: Given a vector $\mathbf{w}$ and vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, can we find scalars $\lambda_1, \dots, \lambda_k$, such that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \dots \mathbf{v}_k$?

In $\mathbf{R}^2$ and $\mathbf{R}^3$ we have a geometric notion of vector addition and scalar multiplication. We think of vectors as arrows with a length and a direction.

The parallelogram law says that the sum $\mathbf{u}+\mathbf{v}$ is given by placing the start of $\mathbf{v}$ where $\mathbf{u}$ ends. Check this by drawing $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1+2 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

If $\lambda > 0$ then scalar multiplication $\lambda\,\mathbf{u}$ is the vector in the same direction as $\mathbf{u}$ with length λ times the length of $\mathbf{u}$. Check this by drawing $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $2\mathbf{u} = \begin{bmatrix} 2\cdot 1 \\ 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Ex Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Express $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Sol. We start by drawing a net of parallelograms with sides $\mathbf{v}_1$ and $\mathbf{v}_2$. Then we see how far we should go first in the $\mathbf{v}_1$ and then in the $\mathbf{v}_2$ direction to reach $\mathbf{b}$. We see that $\mathbf{b} = \mathbf{v}_1 + 2\mathbf{v}_2$.

Ex Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Express $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Sol. Since $\mathbf{v}_2 = 2\mathbf{v}_1$ we can only reach vectors $\mathbf{b}$ which are on the line $t\mathbf{v}_1$ for some t, but this can not be equal to be for any t since one of the components of $\mathbf{b}$ vanishes.

Ex Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.
Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.
Sol $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ if we can find scalars x_1, x_2, x_3 so

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

If we write it out we get the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}.$$

If we add up the vectors to the left we get $\begin{bmatrix} x_1 + 4x_2 + 3x_3 \\ 2x_2 + 6x_3 \\ 3x_1 + 14x_2 + 10x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

i.e. we get a linear system of equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}.$$

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 2 & 1 & -2 \end{bmatrix} \; (3) - 3(1) \quad \Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{bmatrix} \; (2)/2 \; \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -10 \end{bmatrix} \; (3) - 2(2) \quad \Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \; (3)/(-5)$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \; (1) - 4(2) \quad \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \; \begin{matrix} (1) + 9(3) \\ (2) - 3(3) \end{matrix}$$

i.e. we get the system

$$\begin{aligned} x_1 \qquad &= 1 \\ x_2 \quad &= -2 \\ x_3 &= 2 \end{aligned}$$

and hence

$$\mathbf{b} = \mathbf{a}_1 - 2\mathbf{a}_2 + 2\mathbf{a}_3$$

Note that $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix $\begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \, \mathbf{a}_3 \, \mathbf{b} \end{bmatrix}$. Hence solving the vector equation $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ is the same as solving the linear system with augmented matrix $\begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \, \mathbf{a}_3 \, \mathbf{b} \end{bmatrix}$.

In general the vector equation

$$\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_k\mathbf{a}_k$$

has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_k & \mathbf{b} \end{bmatrix}$$

i.e. $\mathbf{b}$ can be generated as a linear combination of $\mathbf{a}_1, \cdots, \mathbf{a}_k$ if and only if there is a solution to the linear system with the corresponding augmented matrix.

Matrix Multiplication

Recall that the dot product of two vectors $\mathbf{w} = \begin{bmatrix} w_1\, w_2 \dots w_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is

$$\mathbf{w} \cdot \mathbf{v} = w_1 v_1 + w_2 v_2 + \cdots + w_n v_n$$

One can think of linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \tag{3.1}$$

as a single vector equation in matrix form

$$A\mathbf{x} = \mathbf{b}, \tag{3.2}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{in} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and **matrix multiplication** $A\mathbf{x}$ of the $m \times n$ matrix A and the $n \times 1$ column vector $\mathbf{x}$ is defined to be the $m \times 1$ column vector formed from the dot product of the row vectors $\mathbf{w}_i = \begin{bmatrix} a_{i1}\, a_{i2} \dots a_{in} \end{bmatrix}$ of A with the column vector $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 \cdot \mathbf{x} \\ \mathbf{w}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{w}_m \cdot \mathbf{x} \end{bmatrix}, \qquad \text{if} \quad A = \begin{bmatrix} - & \mathbf{w}_1 & - \\ - & \mathbf{w}_2 & - \\ & \vdots & \\ - & \mathbf{w}_m & - \end{bmatrix} \tag{3.3}$$

Then (3.2) says that the column vector $A\mathbf{x}$ is equal to the column vector $\mathbf{b}$.
(3.1) just says that the components of these column vectors are equal.

The product (3.3) can be written as linear combination of the column vectors $\mathbf{v}_i$ of A

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_m, \quad \text{if} \quad A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ | & | & & | \end{bmatrix} \tag{3.4}$$

(3.4) is multiplication by columns. (3.3) is multiplication by rows.

Multiplying by the $m \times n$ matrix A hence defines a map $\mathbf{f} : \mathbf{R}^n \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^m$,
for each $n \times 1$ column vector $\mathbf{x}$ we get an $m \times 1$ column vector $A\mathbf{x}$ defined by (3.3).
The map is linear; $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$. All linear maps are of this form.

1.5 in Lay et al Solution sets of linear systems

A **homogeneous system** is a system of the form

$$A\mathbf{x} = \mathbf{0}$$

where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbf{R}^m$.
A homogeneous system always has the **trivial solution** $\mathbf{x} = \mathbf{0}$ so its consistent.
Consistent systems with a free variable have infinitely many solutions.
A homogeneous system has a **nontrivial solution** $\mathbf{x} \neq \mathbf{0}$ if and only if it has free variables.

Ex 1 Describe the solution set of the system $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 4x_1 + 8x_2 - 11x_3 &= 0 \end{aligned}$

Sol There is at least one free variable since there are 2 equations in 3 variables.

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -11 & 0 \end{bmatrix} \sim \begin{matrix} \\ -4(1) \end{matrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{matrix} +3(1) \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 &= 0 \end{aligned} \qquad \Leftrightarrow \qquad x_1 = -2x_2, \quad x_2 = \text{free}, \quad x_3 = 0$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2 \mathbf{v}$$

This is the parametric equation of a line through $\mathbf{0}$ in the direction of $\mathbf{v}$

Ex 2 Determine the solution set of $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 4x_1 + 8x_2 - 11x_3 &= 2 \end{aligned}$

Sol $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -11 & 2 \end{bmatrix} \sim \begin{matrix} \\ -4(1) \end{matrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{matrix} +3(1) \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 \\ x_2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}, \quad x_2 \text{ is a free parameter.}$$

This is the parametric equation of a line through $\mathbf{p}$ in the direction of $\mathbf{v}$, parallel to Ex 1.

If the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is consistent its solution set is parallel to the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Th Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and let $\mathbf{p}$ be a solution. Then any other solution $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is a solution to $A\mathbf{v}_h = \mathbf{0}$.
Pf Since matrix multiplication is linear $A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b}$.

Ex Describe the solution set to $x_1 - 2x_2 - 2x_3 = b$ for $b = 0, 1$.
Sol x_2 and x_3 are free variables

$$\mathbf{x} = \begin{bmatrix} b + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}_1 + x_3 \mathbf{v}_2$$

Since x_2 and x_3 are free parameters this is the **parametric vector equation** of a plane through $\mathbf{p}$ and parallel to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
If $b{=}0$ its the plane spanned by $\mathbf{v}_1, \mathbf{v}_2$ and if $b{\neq}0$ it is a plane parallel to this plane.

Summary and Conceptual Questions

We can now write a linear system with augmented matrix

$$\left[\begin{array}{rrr|r} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{array}\right], \tag{3.5}$$

as a **System of Linear Equations**

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 9 \\ -3x_1 + x_2 \qquad &= -2 \end{aligned} \tag{3.6}$$

as a **Vector Equation**

$$x_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix} \tag{3.7}$$

or as a **Matrix Equation**

$$\begin{bmatrix} 2 & 3 & 4 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

Viewing the system as the intersection of planes (3.6) is called the **row picture** since each equation corresponds to a row of the augmented matrix (3.5).
Viewing the system as a linear combination of vectors (3.7) is called the **column picture** since each vector corresponds to a column of the augmented matrix (3.5).

Matrix multiplication can be calculated in two ways corresponding to the row picture:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 1\cdot 7 + (-4)\cdot(-6) \\ 3\cdot 7 + \ 2\cdot(-6) \\ 0\cdot 7 + 5\cdot(-6) \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}.$$

respective;y the column picture:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}.$$

An $n{\times}n$ matrix A gives a linear map $\mathbb{R}^n \ni \mathbf{x} \to A\mathbf{x} \in \mathbb{R}^n$; i.e. $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, $A(\lambda\mathbf{x}) = \lambda A(\mathbf{x})$.

If A is an $n \times n$ matrix, we have learned that the linear system

$$A\mathbf{x} = \mathbf{b}$$

can be solved uniquely for every $\mathbf{b}$ iff the reduced row echelon form has only 1's in the diagonal.

An $n{\times}n$ matrix B is called an **inverse** of an $n{\times}n$ A if the solution to $A\mathbf{x} = \mathbf{b}$ can be give as

$$\mathbf{x} = B\mathbf{b}$$

Question When does an $n \times n$ matrix A have an inverse? Hint: Try the 2×2 case.

The system $A\mathbf{x} = \mathbf{0}$ is called the **homogeneous equation**:

$$A\mathbf{x} = \mathbf{0}$$

It always has the **trivial** solution $\mathbf{x} = \mathbf{0}$. A solution $\mathbf{x} \neq \mathbf{0}$ is called **nontrivial**.

Question Let A be an $n{\times}n$ matrix. When does the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution?