5. Lecture 5: 2.2 Linear Transformations in Geometry

Draw the effect of the following transformations on the square $\{(x_1, x_2);\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$

Ex 1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ **rotates** vectors an angle $\pi/2$ counterclockwise.

Ex 2 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ **scales** vectors by a factor 3.

Ex 3 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ **projects** vectors onto the x_1 axis.

Ex 4 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ **reflects** vectors in the x_1 axis.

Ex 5 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the **identity** map with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Ex 6 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ x_2 \end{bmatrix}$ is called **shear**

Orthogonal Projections

Let L be a line in the plane going through the origin. Any vector can be written uniquely as

$$\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp},$$

where $\mathbf{x}^{\parallel}$ is parallel to L and $\mathbf{x}^{\perp}$ is perpendicular to the line L. The transformation

$$\text{proj}_L(\mathbf{x}) = \mathbf{x}^{\parallel}$$

is called the **orthogonal projection** of $\mathbf{x}$ onto L. Let $\mathbf{u}$ be a **unit vector** parallel to L. (Unit vector means it has length 1 so $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u} = 1$. If $\mathbf{w}$ is any vector one obtain a unit vector in the same direction by $\mathbf{u} = \mathbf{w}/\|\mathbf{w}\|$.) Then

$$\mathbf{x}^{\parallel} = k\,\mathbf{u},$$

for some scalar k to be determined. Now $\mathbf{x}^{\perp} = \mathbf{x} - \mathbf{x}^{\parallel} = \mathbf{x} - k\mathbf{u}$ is perpendicular to $\mathbf{u}$ so

$$(\mathbf{x} - k\mathbf{u})\cdot\mathbf{u} = 0.$$

It follows that $\mathbf{x}\cdot\mathbf{u} - k\,\mathbf{u}\cdot\mathbf{u} = 0$ and since $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\| = 1$ we have $k = \mathbf{x}\cdot\mathbf{u}$. We conclude that

$$\text{proj}_L(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{u})\,\mathbf{u}.$$

Writing out the dot product above in components

$$\mathbf{x}\cdot\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = x_1u_1 + x_2u_2,$$

the right hand side becomes

$$(x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = x_1u_1\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + x_2u_2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = x_1\begin{bmatrix} u_1u_1 \\ u_1u_2 \end{bmatrix} + x_2\begin{bmatrix} u_2u_1 \\ u_2u_2 \end{bmatrix} = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence

$$\text{proj}_L(\mathbf{x}) = P\mathbf{x}, \qquad \text{where} \qquad P = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$$

Reflections

The reflection in the line L is defined to be

$$\operatorname{ref}_L(\mathbf{x}) = \mathbf{x}^{\parallel} - \mathbf{x}^{\perp}$$

This can be written as

$$\operatorname{ref}_L(\mathbf{x}) = 2\mathbf{x}^{\parallel} - (\mathbf{x}^{\perp} + \mathbf{x}^{\parallel}) = 2\operatorname{proj}_L(\mathbf{x}) - \mathbf{x}$$

Hence using that we already derived the matrix P for the projection

$$\operatorname{ref}_L(\mathbf{x}) = S\mathbf{x}, \qquad \text{where} \quad S = 2P - I, \quad \text{and} \quad P = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix},$$

where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector in the direction of L. Here

$$S = 2P - I = \begin{bmatrix} 2u_1^2 - 1 & 2u_2u_1 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Since $u_1^2 + u_2^2 = \|\mathbf{u}\|^2 = 1$ it follows that $2u_1^2 - 1 = -(2u_2^2 - 1)$ and hence for some a and b

$$S = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \tag{5.1}$$

Moreover a calculation shows that $a^2+b^2 = (2u_1^2-1)(-(2u_2^2-1))+(2u_1u_2)^2 = 2u_1^2+2u_2^2-1=1$. It is not surprising that the length of the column vectors is 1 since they are the image of $\mathbf{e}_1$ respectively $\mathbf{e}_2$ and reflection preserves the lengths, i.e. $\|S\mathbf{x}\| = \|\mathbf{x}\|$.

There is a different way to show that (5.1) represents a reflection outlined in Problem 17 of the text book. By Example 2 of Lecture 4, to determine a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ it is enough to say what its image of two vectors that are not parallel are. We can e.g. take $\mathbf{u}$ to be a vector in the direction of the line and $\mathbf{v}$ to be a vector perpendicular to the line. Then $T(\mathbf{u}) = \mathbf{u}$ since a vector in the line stays the same and $T(\mathbf{v}) = -\mathbf{v}$ since a vector perpendicular to the line gets reflected to its negative. These correspond to systems of equations

$$(S - I)\mathbf{u} = \mathbf{0}, \qquad \text{and} \qquad (S + I)\mathbf{v} = \mathbf{0},$$

or in components

$$\begin{bmatrix} a-1 & b \\ b & -a-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} a+1 & b \\ b & -a+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that can be solved

$$\mathbf{u} = \begin{bmatrix} b \\ 1-a \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$$

Rotations

The map $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\to\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-x_2\\x_1\end{bmatrix}=\begin{bmatrix}y_1\\y_2\end{bmatrix}=\mathbf{y}$ rotates vectors $\pi/2$ counterclockwise.

In fact the condition that $\mathbf{y}$ is perpendicular to $\mathbf{x}$:

$$\mathbf{y}\cdot\mathbf{x}=y_1x_1+y_2x_2=0$$

is equivalent to that

$$\mathbf{y}=k\begin{bmatrix}-x_2\\x_1\end{bmatrix}$$

for some constant k. And the condition that

$$\|\mathbf{y}\|=|k|\sqrt{(-x_2)^2+x_1^2}=\|\mathbf{x}\|$$

is equivalent to $|k|=1$. The choice of $k=1$ corresponds to rotation by $\pi/2$ instead of $-\pi/2$.

Now let $T(\mathbf{x})$ be the transformation that rotates the vector $\mathbf{x}$ an angle θ counterclockwise. If we decompose the vector $T(\mathbf{x})$ into one part that is parallel to $\mathbf{x}$ and one part that is perpendicular we get

$$T(\mathbf{x})=\cos\theta\,\mathbf{x}+\sin\theta\,\mathbf{y}=\cos\theta\begin{bmatrix}x_1\\x_2\end{bmatrix}+\sin\theta\begin{bmatrix}-x_2\\x_1\end{bmatrix}=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Orthogonal transformations

Rotations and Reflections are both part of a larger group of linear transformations called **orthogonal transformations**. Those are linear transformations $T(\mathbf{u})=Q\mathbf{u}$ that satisfy

$$(Q\mathbf{u})\cdot(Q\mathbf{v})=\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\phi,$$

where ϕ is the angle between $\mathbf{u}$ and $\mathbf{v}$. It is easy to see that the columns of the matrices Q have to be orthonormal, i.e. orthogonal and have length 1 (normal here refers to normalized to have length, also called norm, equal to 1). If $\mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$ then $Q\mathbf{e}_1$ and $Q\mathbf{e}_2$ are the columns of the matrix Q and for $i,j=1,2$

$$(Q\mathbf{e}_i)\cdot(Q\mathbf{e}j)=\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij},\qquad\text{where}\qquad\delta_{ij}=\begin{cases}1, & \text{if } i=j,\\0, & \text{if } i\neq j,\end{cases},$$

and that proves that the matrix S corresponds to the reflection in the line L if $a^2+b^2=1$.

Scaling combined with a Rotation

A scaling is the simplest transformation that just multiplies a vector with a scalar $S(\mathbf{x})=r\mathbf{x}$.

A rotation $T(\mathbf{x})$ combined with a scaling $S(\mathbf{x})=r\mathbf{x}$ is the combined map obtained by first rotating the vector an angle θ and then multiplying the resulting vector by a the scalar r.

$$\mathbf{x}\to r\,T(\mathbf{x})=r\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}r\cos\theta&-r\sin\theta\\r\sin\theta&r\cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

Summary and conceptual questions

Let L be a line in the plane going through the origin. Any vector can be written uniquely as

$$\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp},$$

where $\mathbf{x}^{\parallel}$ is parallel to L and $\mathbf{x}^{\perp}$ is perpendicular to the line L. The transformation

$$\text{proj}_L(\mathbf{x}) = \mathbf{x}^{\parallel}$$

is called the **orthogonal projection** of $\mathbf{x}$ onto L. Let $\mathbf{u}$ be a unit vector parallel to L. (If $\mathbf{w}$ is any vector one obtains a unit vector in the same direction by $\mathbf{u} = \mathbf{w}/\|\mathbf{w}\|$.) Then

$$\text{proj}_L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\, \mathbf{u}.$$

In fact

$$\mathbf{x}^{\parallel} = k\, \mathbf{u}$$

and k is determined by that This follows from that

$$0 = \mathbf{x}^{\perp} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} - \mathbf{x}^{\parallel} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} - k\, \mathbf{u} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} - k$$

Let $T(\mathbf{x})$ be the transformation that rotates the vector $\mathbf{x}$ an angle θ counterclockwise.
If we decompose the vector $T(\mathbf{x})$ into one part that is parallel to $\mathbf{x}$ and one part that is perpendicular we get

$$T(\mathbf{x}) = \cos\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sin\theta \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Rotations and Reflections are both part of a larger group of linear transformations called **orthogonal transformations** $T(\mathbf{u}) = Q\mathbf{u}$ that satisfy

$$(Q\mathbf{u}) \cdot (Q\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\phi,$$

where ϕ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Question Which of the transformations discussed in this section preserve the length of vectors (and angle between vectors)?

Question Which of the transformations discussed in this section are invertible?

Question Which of the transformations discussed in this section preserve the area?
(For which is the area of the image of the square the same as the area of the square?)