7. Lecture 7: 2.4 The inverse of a matrix

Def A transformation $T\!:\!X\!\to\!Y$ is called **invertible** if the equation $T(\mathbf{x}) = \mathbf{y}$ has a unique solution $\mathbf{x}\in X$ for each $\mathbf{y}\in Y$. An **inverse** S to T is a map such that $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{y}))=\mathbf{y}$ for all $\mathbf{x}\in X$ and $\mathbf{y}\in Y$. T is invertible if and only if it has an inverse denoted T^{-1}.

Def An $n\times n$ matrix is called **invertible** if the linear transformation $T(\mathbf{x})=A\mathbf{x}$ is invertible. In Lecture 4 we saw that then the inverse is also a linear map. We denote the $n\times n$ matrix of the inverse linear map by A^{-1}. It hence satisfies $T^{-1}(\mathbf{y}) = A^{-1}\mathbf{y}$.

Th The $n \times n$ matrix A is invertible if and only if there is a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I, \tag{7.1}$$

where I is the identity matrix.
Pf If $T(\mathbf{x})=A\mathbf{x}$ is invertible then the equation $T^{-1}(T(\mathbf{x}))=\mathbf{x}$ is equivalent to $A^{-1}A\mathbf{x}=\mathbf{x}=I\mathbf{x}$. However if this holds for all $\mathbf{x}$ it follows that $A^{-1}A=I$. Similarly one proves that $AA^{-1}=I$. On the other hand suppose now that $T(\mathbf{x})=A\mathbf{x}$ and there is a matrix A^{-1} satisfying (7.1). We claim that $\mathbf{x} = A^{-1}\mathbf{y}$ is a unique solution to $A\mathbf{x} = \mathbf{y}$ for every $\mathbf{y}$. In fact, it is a solution since $AA^{-1}\mathbf{y} = I\mathbf{y} = \mathbf{y}$, and the solution is unique since $A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x}$.

We could alternatively have taken the existence of a matrix A^{-1} satisfying (7.1) as the definition of invertibility and inverse of A.

Th Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc\neq 0$ then A is invertible and $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $ad-bc=0$ then A is not invertible.
Pf If $ad-bc\neq 0$ its easy to check that $AA^{-1} = A^{-1}A = I$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}.$$

If $ad-bc=0$ then (a,b) and (c,d) are proportional and the system

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned}$$

does not have a unique solution so A is not invertible.

Ex Solve the system

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

$$A=\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix},\ A^{-1}=\frac{1}{7\cdot 2-3\cdot 5}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix}=\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

Th $(BA)^{-1}=A^{-1}B^{-1}$ and $(A^{-1})^{-1}=A$.
Pf This is clear from the composition of transformations, or from the associative property of matrix multiplication: $A^{-1}B^{-1}BA=A^{-1}IA=A^{-1}A=I$ and $BAA^{-1}B^{-1}=BIB^{-1}=I$.

How to calculate the inverse

If we can solve $A\mathbf{x} = \mathbf{y}$ for any $\mathbf{y}$ we will get the inverse $\mathbf{x} = A^{-1}\mathbf{y}$.

Ex Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol We perform row operations to solve the system $A\mathbf{x} = \mathbf{y}$:

$$\begin{cases} x_1 \quad = y_1 \\ -3x_1 + x_3 = y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_3 = 3y_1 + y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_3 \\ x_3 = 3y_1 + y_2 \end{cases}$$

adding three times the first equation to the second and then switching the second and the third equations. The system on the right is $\mathbf{x} = A^{-1}\mathbf{y}$ so we must have that $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$. Its easy to check that $AA^{-1} = I$.

The calculations above can be performed without writing out the variables as row operations directly to the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$;

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim (2) + 3(1) \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{matrix} \\ (3) \\ (2) \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{bmatrix}$$

We have found an algorithm for determining if A is invertible and finding the inverse: Calculate the reduced row echelon form of the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If it is of the form $\begin{bmatrix} I & B \end{bmatrix}$ then A is invertible and $A^{-1} = B$. Otherwise A is not invertible.

One can also prove that this works multiplying by **elementary** matrices which correspond to elementary row operations. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Multiplying by E_1 adds 3 times row one to row two:

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Multiplying by E_2 switches row two and row three:

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence

$$E_2 E_1 A = I$$

and multiplying both sides by A^{-1} to the right gives since $AA^{-1} = I$ and $IA^{-1} = A^{-1}$:

$$E_2 E_1 I = A^{-1}$$

Hence a sequence of elementary row operations that reduce A to I reduce I to A^{-1}. This argument assumed that A was invertible, but it also follows since each elementary matrix is invertible since the row operations are reversible and hence multiplying by the inverse of the elementary matrices gives $A = E_1^{-1} E_2^{-1}$ so A is invertible since it is a product of invertible.

Geometrical interpretation of the determinant

Th Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc\neq 0$ then A is invertible and $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $ad-bc=0$ then A is not invertible.

Def The **determinant** of the 2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number $\det(A)=ad-bc$.

We have seen from the proof of the above theorem or directly that the determinant is 0 is equivalent to that the columns or rows are parallel. Now we will give a geometric meaning to the magnitude of the determinant.
Let $\mathbf{v}=\begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix} b \\ d \end{bmatrix}$ be the column vectors of A, and let $\mathbf{v}_{\text{rot}}=\begin{bmatrix} -c \\ a \end{bmatrix}$ be the rotation of the vector $\mathbf{v}$ by $\pi/2$ counterclockwise. We see that by the definition of the dot product

$$\det(A)=ad-bc=\mathbf{v}_{\text{rot}}\cdot\mathbf{w}=\|\mathbf{v}_{\text{rot}}\|\,\|\mathbf{w}\|\cos\phi,$$

where ϕ is the angle between $\mathbf{v}_{\text{rot}}$ and $\mathbf{w}$. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$ then $\theta=\pi/2-\phi$ and $\cos(\pi/2-\theta)=\sin\theta$, and $\|\mathbf{v}_{\text{rot}}\|=\|\mathbf{v}\|$, so

$$\det(A)=\|\mathbf{v}\|\,\|\mathbf{w}\|\sin\theta,$$

We conclude that

$$\big|\det(A)\big| = \text{Area of parallelogram with sides } \mathbf{v} \text{ and } \mathbf{w}.$$

Question What is the area of the image of the unit square by the map with matrix A above?

Question What is the inverse of a scaling by a factor 3 and what is its matrix?

The transpose of an orthogonal matrix

The transpose A^T is the matrix with rows and columns interchanged, $(A^T)_{ij}=(A)_{ji}$

Ex If $A=\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & -1 \\ 4 & 5 & 2 \end{bmatrix}$ then $A^T=\begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$.

Def An $n\times n$ matrix

$$Q=\begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$$

is called orthogonal if the column vectors are orthonormal, i.e. for all i,j

$$\mathbf{q}_i\cdot\mathbf{q}_j=\delta_{ij},\qquad \text{where}\qquad \delta_{ij}=\begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j, \end{cases}.$$

An equivalent way to formulate this is that $Q^TQ=I$. This is because the columns of Q become the rows of Q^T and the matrix product is formed by taking the dot product of the rows of Q^T by the columns of Q, by the row column rule.

Question What is the inverse of a rotation counterclockwise $\pi/2$ and what is its matrix?

Summary and Conceptual Questions

Def A transformation $T\colon X\to Y$ is called **invertible** if the equation

$$T(\mathbf{x}) = \mathbf{y}, \quad \text{has a unique solution } \mathbf{x}\in X \text{ for each } \mathbf{y}\in Y.$$

An **inverse** T^{-1} to T is a map such that

$$T^{-1}(T(\mathbf{x}))=\mathbf{x}, \quad \text{and} \quad T(T^{-1}(\mathbf{y}))=\mathbf{y}, \quad \text{for all } \mathbf{x}\in X,\ \mathbf{y}\in Y. \tag{7.2}$$

T is invertible if and only if it has an inverse T^{-1}.
Def A matrix A is called **invertible** if the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is invertible. The inverse is also linear and we denote its matrix by A^{-1} so $T^{-1}(\mathbf{y})=A^{-1}\mathbf{y}$.

Th An $n \times n$ matrix A is invertible if and only if there is an $n \times n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I, \qquad \text{where } I \text{ is the identity matrix.} \tag{7.3}$$

In fact (7.2) is equivalent to $A^{-1}A\mathbf{x}=\mathbf{x}$ and $AA^{-1}\mathbf{y}=\mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ which is equivalent to (7.3). We could alternatively have taken the existence of a matrix A^{-1} satisfying (7.3) as the definition of invertibility, instead of the map $T(\mathbf{x}) = A\mathbf{x}$ having an inverse satisfying (7.2).

Th Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc\neq 0$ then A is invertible and $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

To find the inverse of A we perform row operations to solve the system $A\mathbf{x}=\mathbf{y}$ for $\mathbf{x}=B\mathbf{y}$. Then $A^{-1}=B$.

Ex Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol

$$\begin{cases} x_1 = y_1 \\ -3x_1 + x_3 = y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_3 = 3y_1 + y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_3 \\ x_3 = 3y_1 + y_2 \end{cases}$$

adding three times the first equation to the second and then switching the second and third. The system on the right is $\mathbf{x} = A^{-1}\mathbf{y}$ so we must have that

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$. Its easy to check that $AA^{-1} = I$.

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$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim (2)+3(1) \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{matrix} \\ (3) \\ (2) \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{bmatrix}$$

We have found an algorithm for determining if A is invertible and finding the inverse:
Calculate the reduced row echelon form of the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$.
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